Global Approximate Output Tracking for Nonlinear Systems

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Abstract—This paper addresses the global output tracking problem for nonlinear systems with singular points. For nonlinear systems which satisfy a suitable observability condition, the authors identify a class of smooth output trajectories which the system can track using continuous open-loop controls. This class includes all output trajectories generated by smooth state feedback. They then study the problem of approximate output tracking using discontinuous time-varying feedback controllers. Given a smooth output trajectory for which exact tracking is possible, the authors construct a discontinuous feedback controller which achieves robust tracking of the desired output trajectory in the face of perturbations. Finally, it is shown that their results can be applied to the control of a chain system, and some numerical results are presented to illustrate the performance of their controller.

Index Terms—Discontinuous state feedback, global output tracking, nonlinear systems, singular points.

I. INTRODUCTION

The output tracking problem has a long-standing history; see, e.g., [5], [24], [25], [27], and the references therein. One attractive feature of this problem is that the global linear state-space methods (c.f. [25]) can be extended to the nonlinear case with only a slight increase in complexity. Unfortunately, for most cases one can achieve only local results because of the local nature of the inversion algorithm in the nonlinear case (c.f. [8], [13], [14], [21], [22], and [23]).

Typically, the obstruction to achieving global results stems from the existence of so-called singular submanifolds, where the decoupling matrix [17], [19] loses rank and, as a result, the input-output map of the system is no longer one-to-one.

There has been some work on the global output tracking problem. In [15] and [16] the notion of the degree of singularity of an output function at time \( t \) is introduced for single-input/single-output (SISO) affine nonlinear systems. The degree of singularity is related to the degree of tangency of the state trajectory when it enters the singular submanifold.

In [15] it is assumed that the initial state belongs to the singular submanifold, but that the trajectories of the system never return to it; in [16] a class of output functions is identified for which global output tracking can be achieved. Roughly speaking, the output trajectories which can be tracked must satisfy a certain number of restrictions, which in turn ensure that the trajectories of the system are transversal to the singular submanifold in some suitable sense. In [7] the problem of exact tracking is studied using results on singular ordinary differential equations. In particular, for real analytic affine systems with a single input and output the notion of rank of a singularity is introduced, and results on the multiplicity of solutions are presented. Conditions under which the singular tracking control is smooth or analytic are given in [18], assuming that the inputs and some of their derivatives are related to the outputs and their derivatives via a singular ordinary differential equation. This has connections to this work in that our observability assumption implies the existence of such a differential equation.

Even though the results in [7], [15], [16], and [18] are attractive from the system theoretic viewpoint, to track an admissible output trajectory exactly requires that the initial state is known, a perfect model is available, and the system is affected by no disturbances. In this paper we shall generalize and improve the results presented in [15] and [16] in different ways. First, we shall consider the multi-input/multi-output (MIMO) case, as opposed to [15] and [16] where only the SISO case was considered. Second, we shall relax the observability assumption made in [16] in the sense that it does not need to be satisfied globally. Finally, we shall construct a time-varying discontinuous controller which achieves robust approximate output tracking, in the sense that we define below.

The rest of the paper is organized as follows. In Section II we introduce our basic assumptions on local invertibility and partial observability. In Section III we give a classification for the singularities which can appear in the nonlinear inversion algorithm. In Section IV we study global exact tracking in the presence of singularities and identify smooth output functions which can be tracked exactly using continuous controls. In Section V we state and prove our main results on asymptotic output tracking using a discontinuous feedback controller. In Section VI we present an application of our results to the output tracking for chained systems. Finally, some concluding remarks are offered in Section VII.

II. PRELIMINARIES AND BASIC ASSUMPTIONS

Consider the square nonlinear control system

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} w_i g_i(x) = f(x) + g(x)u \\
y &= h(x)
\end{align*}
\]

(1)

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where the state $x \in M$, an $n$-dimensional smooth manifold, the control $u = [u_1, \ldots, u_m] \in \mathbb{R}^m$, and the output $y \in \mathbb{R}^n$. Throughout the paper we assume that the output map $h : M \to \mathbb{R}^n$ and the vector fields $f(x), g_i(x)$ are smooth. Unless stated otherwise, the inputs or controls $u(t)$ are assumed to be piecewise smooth functions of time on a closed interval $[t_0, t_1]$.

Observe that the solution of the differential equation (1) may not be defined on all of $[0, \infty)$ because of possible finite escape times. Therefore, we explicitly assume that for every piecewise smooth control $u : [0, \infty) \to \mathbb{R}^m$, the system (1) has a solution defined on some maximal interval which includes $[t_0, t_1]$.

Given a smooth function $k : M \to \mathbb{R}$ and a vector field $X(x)$ on $M$, $L_Xk(x)$ denotes the Lie derivative of $k(x)$ along $X(x)$: that is, $L_Xk(x) = \langle dk(x), X(x) \rangle$. If $X(x) = [X_1(x), \ldots, X_m(x)]$ is a matrix whose columns are vector fields, then $L_Xk(x)$ denotes the row vector $[L_XX_1(x), \ldots, L_XX_m(x)]$.

The relative degrees of (1) with respect to the output map $h$ are the smallest integers $r_1, \ldots, r_m$ such that, for

- $L_{h_j}h_i(x) = 0$, for $k = 0, \ldots, r_j - 2$ and for all $x \in M$.
- $L_{h_j}^{r_j-1}h_i(x) \neq 0$, for some $x \in M$.

If $L_{h_j}^{r_j}h_i(x) = 0$, for $k \geq 0$ and for all $x \in M$, set $r_j = \infty$.

Given a map $h : M \to \mathbb{R}^m$ with finite relative degrees $r_1, \ldots, r_m$, we introduce the $m \times m$ matrix $a(x)$ defined by $a(x) = [L_{h_j}^{r_j}h_i(x)]$ and the $m \times m$ matrix $b(x)$ defined by $b(x) = [L_{h_j}^{r_j-1}h_i(x)]$. The matrix $b(x)$ is often called the decoupling matrix of the system $\Sigma$ [17], [19]. Given a point $x_0 \in M$, if there exists a neighborhood $U_0$ of $x_0$ such that $\text{rank} \ b(x) = m, \forall x \in U_0$, then the system $\Sigma$ is said to have a well-defined vector relative degree at $x_0$.

Set $[x] = r_1 + r_2 + \cdots + r_m$ and define a map $h^F : M \to \mathbb{R}^m$ by

$$x \mapsto (h_1(x), \ldots, L_{r_1}h_1(x), \ldots, h_m(x), \ldots, L_{r_m}h_m(x)).$$

Similarly, if $y(t)$ is a sufficiently smooth output function for (1) we set

$$y^F(t) = (y_1(t), y_1^{(1)}(t), \ldots, y_1^{(r_1-1)}(t), \ldots, y_m(t), \ldots, y_m^{(r_m-1)}(t)).$$

so that

$$y^F(t) = h^F(x(t))$$

$$y^F(t) = a(x(t)) + b(x(t))u(t).$$

For those points $x \in M$ where the matrix $b(x)$ is invertible, one can solve for $u$ as a function of $x$ and $y^F$ using (4). The set of points where $b(x)$ is not invertible will be referred to as the singular set. The singular set is defined as $M_s = \{x \in M \mid \det b(x) = 0\}$. Throughout the paper we assume that the system $\Sigma$ has, locally, a well-defined vector relative degree.

### Assumption A1: $\max_{x \in M_s} \text{rank} \ b(x) = m$.

The image of $M$ (respectively, $M_s$) under the map $h^F$ is a subset of $\mathbb{R}^m$ which we denote by $N$ (respectively, $N_s$). Our second assumption is that the system $\Sigma$ is partially observable, to the extent that the matrices $a(x)$ and $b(x)$ in (4) can be expressed in terms of $y$ and its derivatives. More precisely, we will make the following assumption.

### Assumption A2: If $M_s \neq \emptyset$, then there exist smooth maps $A : N \setminus N_s \to \mathbb{R}^m$, $B : N \setminus N_s \to \mathbb{R}^{m \times m}$ such that on $M \setminus M_s$

$$a(x) = A \circ h^F(x)$$

$$b(x) = B \circ h^F(x).$$

### Remark 2.1: Assumption A2 is similar to, but weaker than, the strong observability assumption introduced in [16] for the SISO case; actually, it is stronger than we really need but is adopted to simplify the notation. It suffices to define the maps $A$ and $B$ on a subset $h^F(U \setminus M_s)$ of $N \setminus N_s$ where $U$ is any open subset of $M$ containing $M_s$.

### Remark 2.2: While Assumption A1 is invariant under regular state feedback, in some cases we can use a preliminary state feedback of the form $u = a(x) + v$ to achieve Assumption A2. Consider for example the nonlinear system

$$\dot{x}_1 = x_2^2 + x_2u$$

$$\dot{x}_2 = x_1x_2 + x_1u$$

$$y = x_1 + x_2.$$ 

Here $\dot{y} = x_2(x_1 + x_2) + (x_1 + x_2)u$ hence $a(x) = x_2(x_1 + x_2)$, $b(x) = x_1 + x_2$, and $h^F = x_1 + x_2$. Thus $b(x)$ is a function of $h^F$ but $a(x)$ is not. Nonetheless A2 holds after applying the feedback $u = v = x_2$.

### III. THE DEGREE OF SINGULARITY OF AN OUTPUT FUNCTION

The purpose of this section is to give a precise classification of the singularities that may appear in the output tracking problem.

### Definition 3.1: A mapping $y_d : [t_0, t_1] \to \mathbb{R}^n$ is said to be tracked by the output of (1) on $[t_0, t_1]$ if, for some piecewise smooth control $u : [t_0, t_1] \to \mathbb{R}^m$, the resulting output $y(t) = y_d(t)$ for all $t \in [t_0, t_1]$.

Suppose that $y$ is a smooth mapping tracked by the output of (1) on $[t_0, t_1]$ when the input $u$ is applied, and let $x$ denote the corresponding state trajectory. Then for all $t$ such that $x(t) \in M \setminus M_s$ we know that $\det b(x(t)) \neq 0$ and thus we can solve (4) for $u$ as a function of $x$ and $y^F$; that is

$$u(t) = \frac{1}{\det B(y^F(t))} \text{Adj}[B(y^F(t))] [y^F(t) - A(y^F(t))].$$

If A2 holds, then

$$u(t) = [\text{Adj}(y^F(t))]^{-1} [y^F(t) - A(y^F(t))].$$

which is a well-defined smooth function of $t$ when $x(t) \notin M_s$. Since $B^{-1}(z) = (1/\det B(z)) \text{Adj}(B(z))$ we see that

$$u(t) = \frac{1}{\det B(y^F(t))} \text{Adj}[B(y^F(t))] [y^F(t) - A(y^F(t))].$$

= $(1/\beta y(t)) F_y(t)$
where we set
\[
\begin{align*}
\beta_g(t) &= \text{det } B(y(t)); \\
F_y(t) &= \text{Ad}[B(y(t))][y(t) - A(y(t))],
\end{align*}
\] (9)

It follows that \(x(t) \in M \setminus M_s\) if and only if \(\beta_g(t) \neq 0\), \(x(t_s) \in M_s\) if and only if \(\lim_{t \to t_s} \beta_g(t) = 0\), and \(\beta_g(t)u(t) = F_y(t)\) for all \(t\) such that \(\beta_g(t) \neq 0\). In particular, for all \(t\) such that \(x(t) \in M \setminus M_s\) (or equivalently \(t \in \{t \mid \beta_g(t) \neq 0\}\)) we know that \(u, \beta_g, \) and the components of \(F_y\) are smooth functions of \(t\). Thus \(\beta_g(t)u(t) = F_y(t)\) is infinitely differentiable, and we have
\[
\begin{align*}
\beta_g(t)u(t) &= F_y(t) \\
\beta_g(t)u(t) + \beta_g(t)\dot{u}(t) &= \dot{F}_y(t) \\
&= \sum_{j=0}^\ell \left(\begin{array}{c} \ell \\ j \end{array}\right) \beta_g^{(j)}(t)u^{(\ell-j)}(t) = F_y^{(\ell)}(t)
\end{align*}
\] (10)

for all \(t\) such that \(\beta_g(t) \neq 0\).

We now study what happens when \(x(t)\) approaches \(M_s\). Suppose that \(x(t_s) \in M_s\) but, for \(t \neq t_s\), \(x(t) \not\in M_s\) for \(t\) in some open interval containing \(t_s\). In this case \(\lim_{t \to t_s} \beta_g(t) = 0\) and, since (10) holds when \(t \neq t_s\), we can conclude that
\[
\lim_{t \to t_s} \beta_g(t)u(t) = \lim_{t \to t_s} F_y(t).
\] (11)

If \(u\) is continuous when \(t = t_s\) then
\[
\lim_{t \to t_s} \beta_g(t)u(t) = \lim_{t \to t_s} F_y(t).
\] (12)

Since \(\lim_{t \to t_s} \beta_g(t) = 0\) it follows that \(\lim_{t \to t_s} F_y(t) = 0\).

In a similar manner we can take limits for the identity
\[
\beta_g(t)u(t) + \beta_g(t)\dot{u}(t) = \dot{F}_y(t).
\]
We know that from \(M_s\) we have \(\beta_g(t) = \text{det } B(y(t)) = \text{det } b(x(t))\) and thus \(\frac{\partial}{\partial t} \beta_g(t) = \frac{\partial}{\partial t} \text{det } b(x(t))\). This implies that \(\lim_{t \to t_s} \beta_g(t)\) exists as long as \(\lim_{t \to t_s} u(t)\) is well defined. In particular, if \(u\) is of class \(C^1\) at \(t = t_s\) we have
\[
\lim_{t \to t_s} \beta_g(t)u(t) + \beta_g(t)\dot{u}(t) = \lim_{t \to t_s} \dot{F}_y(t)
\] (13)

which implies that
\[
\lim_{t \to t_s} \dot{F}_y(t) = \left[\lim_{t \to t_s} \beta_g(t)\right]u(t_s).
\] (14)

This last equation imposes an additional restriction on \(F_y\) when \(\lim_{t \to t_s} \beta_g(t) = 0\), namely that \(\lim_{t \to t_s} \dot{F}_y(t) = 0\). Since \(F_y\) is a function of the output \(y\), the restrictions on \(F_y\) translate into restrictions on the possible output functions. In the case where \(\lim_{t \to t_s} \beta_g^{(k)}(t) = 0, \forall k \geq 0\) we can repeat the above arguments and conclude that \(\lim_{t \to t_s} F_y^{(k)}(t) = 0, \forall k \geq 0\). This results in an infinite number of restrictions on \(y\). The following result shows that, in the real analytic case, this degenerate situation will occur only if the state trajectory \(x(t)\) remains on the singular set for all \(t \geq t_0\).

Theorem 3.2: Consider the nonlinear system (1), where \(f, g, h, M, \) and \(u\) are real analytic and Assumptions A1 and A2 are satisfied. Let \(x(t)\) and \(y(t)\) denote the state and output trajectories which result when a control \(u\) is applied and defined \(T_s = \{t \mid 0 \leq t \leq t_s, x(t) \in M_s\}\). Suppose that \(t_0 \notin T_s \neq \emptyset\) and let \(t_s^1 = \min_{0 \leq t \leq t_s} \{t \mid x(t) \in M_s\}\). Then \(\lim_{t \to t_s} \beta_g^{(j)}(t)\) exists for all \(j \geq 0\) and \(\lim_{t \to t_s} \beta_g^{(j)}(t) \neq 0\) for some \(k > 0\).

Proof: Set \(\dot{d}(t) = \text{det } B(x(t))\). Since \(f, g, h, M, \) and \(u\) are real analytic it follows that \(d(t)\) is real analytic also. Thus for \(t\) in some neighborhood of \(t_s^1\)
\[
d(t) = d(t_s^1) + d(t_s^1)(t - t_s^1) + d(t_s^1)(t - t_s^1)^2/2! + \cdots.
\]

Furthermore, \(\dot{d}^{(j)}(t_s^1) = \lim_{t \to t_s^1} \dot{d}^{(j)}(t)\), the limit as \(t\) approaches \(t_s^1\) from the left. Thus \(\dot{d}^{(j)}(t_s^1) = \lim_{t \to t_s^1} \beta_g^{(j)}(t)\) as a consequence of A2.

Suppose that \(\dot{d}^{(j)}(t_s^1) = 0, \forall k > 0\). Then the power series expansion for \(d(t)\) about \(\hat{t} = t_s^1\) is identically zero, and by real analyticity \(d(t) \equiv 0\). This implies that \(\dot{d}(t_0) = 0, \) or equivalently that \(x(t_0) \in M_s\). This contradicts \(t_0 \notin T_s\). Thus there exists a \(k\) such that \(\dot{d}^{(k)}(t_s^1) = \lim_{t \to t_s^1} \beta_g^{(k)}(t) \neq 0\). This implies that \(\exists \delta > 0\) such that \(d(t) \neq 0\) for \(0 < |t - t_s^1| < \delta\).

In particular this means that the set of singular times \(t_s\) when \(\lim_{t \to t_s} x(t) \in M_s\) will form an isolated set of points in \(R\). As a direct consequence of this and Assumption A2 we have \(\lim_{t \to t_s^1} \beta_g^{(k)}(t) \neq 0\) and \(\lim_{t \to t_s^1} \beta_g^{(k)}(t)\) exists for all \(k \geq 0\).

The previous theorem motivates the following definition.

Definition 3.3: Suppose that \(y : [t_0, t_1] \to \mathbb{R}^m\) is a smooth map and that A1 and A2 hold. Then \(y\) has a singularity of degree \(\alpha\) at time \(t_s\) if \(\lim_{t \to t_s} \beta_g^{(k)}(t) = 0\) for \(0 \leq k < \alpha\) and \(\lim_{t \to t_s} \beta_g^{(k)}(t) \neq 0\). In the case where \(M_s = \emptyset\) we set \(\alpha = 0\).

Let \(\alpha_y(t_s) = \alpha\) denote the degree of the singularity of \(y\) at time \(t_s\). In the case where \(\lim_{t \to t_s} \beta_g^{(k)}(t) = 0, \forall k \geq 0\) we set \(\alpha_y(t_s) = \infty\), and if \(\lim_{t \to t_s} \beta_g^{(k)}(t) \neq 0\) we set \(\alpha_y(t_s) = 0\). Note that inherent in the above definition is the assumption that \(\lim_{t \to t_s} \beta_g^{(k)}(t)\) exists for all \(0 \leq k \leq \alpha_y\), and, if \(y(t) = h(x(t))\), then the set \(T_s\) of singular times \(t_s\) when \(x(t_s) \in M_s\) will form an isolated set of points in \(R\).

Remark 3.4: Note that in the real analytic case \(\alpha_y(t_s) = \infty\), if and only if \(\lim_{t \to t_s} \beta_g(t) = \infty, \forall t \geq t_0\), as a consequence of Theorem 3.2. Furthermore, if \(\alpha_y(t) \neq \infty\) on \([t_0, t_1]\), then \(y\) has some maximal degree of singularity \(\alpha^*\) as a consequence of the compactness of \([t_0, t_1]\).

IV. EXACT OUTPUT TRACKING

In this section we will present our results on exact tracking which, in some sense, generalize the results in [16] to the MIMO case.

Theorem 4.1: Suppose that \(y_d : [t_0, t_1] \to \mathbb{R}^m\) is a smooth map with maximal degree of singularity \(\alpha^*\). Under Assumptions A1 and A2 a necessary condition for \(y_d\) to be tracked by the output of (1) using an input \(u\) which is at least \(\alpha^*\) times differentiable is that the following conditions hold.

1) \(y_d(t_0) = h(x_0)\).
2) If $\alpha = \alpha_{s0}(t_s) > 0$ then, for $0 \leq j < \alpha$, 
\[ \lim_{t \to t_s} F^{(j)}(t) = 0 \quad \text{and} \quad \lim_{t \to t_s} F^{(j)}(t) \exists. \]

Proof: Suppose that $y(t) = y_d(t)$ for all $t \in [t_0, t_1]$ using a control $u$ which is $\alpha$ times differentiable. Since $y(t) = h^T(x(t))$ we see that 1) holds. Suppose that $M_s \neq \emptyset$. If $u \in C^{\alpha}[t_0, t_2]$ then the state trajectory $x(t) \in C^{\alpha}[t_0, t_2]$ and thus, as we argued above, (10) implies that 2) holds. \(\square\)

Theorem 4.2: Suppose that $y_d : [t_0, t_1] \to \mathbb{R}^m$ is a smooth map with maximal degree of singularity $\alpha$. Under Assumptions A1 and A2 a sufficient condition for $y_d$ to be tracked by the output of (1) using a continuous input $u$ is that the following conditions hold.

1) $y_d(t_0) = h^T(x_0)$.
2) If $\alpha = \alpha_{s0}(t_s) > 0$ then, for $0 \leq j < \alpha$, 
\[ \lim_{t \to t_s} F^{(j)}(t) = 0 \quad \text{and} \quad \lim_{t \to t_s} F^{(j)}(t) \exists. \]

Proof: Suppose that $y_d(t)$ satisfies conditions 1) and 2) above. To complete the proof we must find a continuous control open-loop control $u(t)$ which results in the output $y_d(t)$.

We begin with the case where $M_s = \emptyset$. Let $x(t)$ denote the trajectory for system (1) produced by the smooth feedback controller $u_d(t) = [b(x(t))]^{-1}[y_d(t) - a(x(t))]$, which is defined on all of $M$. The resulting output $y$ satisfies $y(t) = a(x(t)) + b(x(t))u_d(x(t))$, as a consequence of (4). Thus $y(t) = y_d(t)$ for $t \geq t_0$. But the lower order derivatives of $y$ and $y_d$ agree at $t = t_0$ as a consequence of condition 1), i.e., $y^{(j)}(t_0) = y_d^{(j)}(t_0)$. Thus by integrating repeatedly we can conclude that $y(t) = y_d(t)$ for $t \geq t_0$ where $y$ is the output resulting from the continuous (in fact smooth) open-loop control $u(t) = u_d(x(t), t)$.

Now suppose that $M_s \neq \emptyset$. We will define an open-loop control law and show that it is continuous and that the resulting output is $y_d(t)$. Let $T_s$ denote the set of singular times $t_s$ for the function $y_d$. From Section III we know that $T_s$ is a set of isolated points. From this and the compactness of $[t_0, t_1]$ we can conclude that $T_s$ has a finite number of elements. By definition, $t_s \in T_s$ iff $\alpha_{s0}(t_s) > 0$. We set $u_d(t) = (1/\beta_{s0}(t_s))F_{g_d}(t)$ if $t \not\in T_s$ and $u_d(t) = \lim_{t \to t_s} F^{(j)}(t)/\lim_{t \to t_s} \beta_{s0}(t)$ if $t \in T_s$, where $\alpha = \alpha_{s0}(t_s) > 0$.

By construction $u_d$ is continuous off the finite set $T_s$. Thus to show that $u_d$ is continuous on all of $T$, we need only verify that, for $t_s \in T_s$, 
\[ u_d(t) = \lim_{t \to t_s} u_d(t). \]

But $y_d(t)$ also solves this differential equation off the finite set $T_s$. Since $y_d(t)$ and $y_d(t)$ are both continuous on all of $[t_0, t_1]$ and $y_d(t) = y_d(t)$ due to condition 1) we see that $y_d(t) = y_d(t)$ for $t \geq t_0$.

Remark 4.3: The above results identify a large class of functions which the output trajectory of system (1) can be made to track through "singular times," but the controller (18) is open loop, and hence will be rather unsatisfactory in practice. One remedy is to use the equivalent closed-loop controller (7). This controller will produce the same state and output trajectories as (18) but, if we are somewhat unsure of our initial conditions or $x(t)$ diverges slightly from $x_d(t)$, this controller can become unbounded. We denote the tracking error and its derivatives by $e_d(t) = y_d(t) - h^T(x(t))$ and $\dot{e}_d(t) = y_d(t) - h^T(x(t))$. Thus we set

\[ e_d(t) = y(t) - y_d(t) \quad \text{and} \quad \dot{e}_d(t) = y_d(t) - y_d(t) \]

V. APPROXIMATE OUTPUT TRACKING

Suppose that $y_d : [t_0, t_1] \to \mathbb{R}^m$ is smooth and satisfies conditions 1) and 2) of Theorem 4.2. In this case we have $y_d(t) = h^T(x(t))$ for some $x_d(t) \in M$. Thus $y_d$ can be tracked exactly by the output of system (1) if the initial state is $x_0$. In practice we must be prepared to deal with modeling errors and external disturbances. In effect we must allow for the situation where $x(t_0) \neq x_0$. In this case the actual output $y$ will not exactly coincide with $y_d$. Unfortunately, our global feedback controller (7), which assumes that we will be able to exactly cancel small terms as $x(t)$ approaches $M_s$, can become unbounded. We denote the tracking error and its derivatives by $e_d(t) = y(t) - y_d(t)$ and $\dot{e}_d(t) = y_d(t) - y_d(t)$.
It is not hard to modify the feedback controller (7) to regulate $e_y$ by output feedback. Consider the controller

$$u^\text{CL}_y(x, t) = [b(x)]^{-1} (v_3(t) + G_{\text{CL}}^o(t) - a(x)) \quad (20)$$

where $G$ can be chosen arbitrarily. Later we will choose $G$ so that $e_y$ is the solution to a stable linear differential equation. This controller is not defined on $M_k$ as the required inverse does not exist. On the other hand, if we were tracking $y_d$ exactly so that $y = y_d$, then $u^\text{CL}_y$ extends to all of $M$ in the sense that, in open-loop form, it is precisely the function $u^\text{OL}_y(t)$ defined by (18). This follows from Theorem 4.2. These considerations lead us to define the discontinuous controller

$$u_x(x, t) = \begin{cases} u^\text{CL}_y(x, t), & \det(b(x)) > \epsilon \\ u^\text{OL}_y(t), & \det(b(x)) \leq \epsilon \end{cases} \quad (21)$$

where $\epsilon > 0$ and $G$ is any $m \times [p]$ matrix. The next theorem shows that, using our discontinuous controller, the nonlinear system (1) has unique solutions and arbitrarily small tracking error for the initial state $x(t_0) = x'_0$ sufficiently close to $x_0$. We will assume that the sets $M_x = \{ x \in M \mid \det(b(x)) = 0 \}$ are embedded submanifolds of $M$.

Theorem 5.1: Suppose that $y_d : [t_0, t_1] \rightarrow \mathbb{R}^m$ satisfies the hypotheses of Theorem 4.2 and thus can be tracked exactly by the output of system (1), provided that $x(t_0) = x_0$. Then, given any $\delta > 0$, there exists $\epsilon > 0$ and an open neighborhood $U_0$ of $x_0$ such that the following holds: if $x(t_0) = x'_0 \in U_0$ then the discontinuous controller $u_x(x, t)$ produces a unique state trajectory for the nonlinear system (1) and an output $y$ which tracks $y_d$ in the sense that $|y_d(t) - g(x(t))| < \delta$ \forall $t \in [t_0, t_1]$. Moreover, along the trajectories of the system (1) the controller $u_x(x, t)$ remains bounded.

Before proving Theorem 5.1 we first establish the following technical lemmas.

Lemma 5.2: Suppose that $y_d : [t_0, t_1] \rightarrow \mathbb{R}^m$ satisfies the hypotheses of Theorem 4.2; $\beta_{\text{CL}}$ is the function defined by (9); $T_s$ is the finite set of singular times; and $t_s \in T_s \cap (t_0, t_1)$. Then, there exists $\epsilon_t > 0$ and $[t_0, t_1] \subset T$ such that $t_s \in (t_0, t_t)$, $|\beta_{\text{CL}}(t_s) - \beta_{\text{CL}}(t)| < \epsilon_t$, and $\beta_{\text{CL}}(t) \neq 0$. If $\beta_{\text{CL}}(t) = 0$ and $\beta_{\text{CL}}(t) \neq 0$, $\beta_{\text{CL}}(t)$ is either increasing or decreasing on $(t_0, t_t)$ and on $(t, t_1)$. In particular, $T_t \cap (t_0, t_1) = \emptyset$.

Proof: If $T_s = \emptyset$, then there is nothing to prove. If $T_s \neq \emptyset$ then $M_s \neq 0$ and Theorem 4.2 implies that, for $u = u^\text{OL}_{y_d}$, the system (1) has state trajectory $\tilde{x}(t)$ and output trajectory $y(t) = y_d(t)$. Set $\beta(t) = \det(b(x(t)))$. By definition $\beta(t)$ is $\beta_{\text{CL}}(t)$ for all $t \not\in T_s$. Suppose that $t_s \in T_s \cap (t_0, t_1)$ and $\alpha \in \alpha_{\text{CL}}(t_s)$. From the definition of $\alpha_{\text{CL}}(t_s)$ it follows that $\lim_{t \rightarrow t_s} \beta_{\text{CL}}(t) = \lim_{t \rightarrow t_s} \det(b(x(t))) = 0$ for all $0 < j < \alpha$ and $\lim_{t \rightarrow t_s} \beta_{\text{CL}}(t) = \lim_{t \rightarrow t_s} \det(b(x(t))) = 0$. We set $\alpha_t = \lim_{t \rightarrow t_s} \beta_{\text{CL}}(t)$ so that, for $t$ near $t_s$, the function $\det(b(x(t)))$ is well approximated by its $\alpha_t$ degree Taylor polynomial $k_{\alpha_t}(t-t_s) \beta_{\text{CL}}(t)$. First we consider the case where $\alpha$ is odd and $k_{\alpha_t} > 0$. This means that $\beta(t)$ is increasing on some open neighborhood $(t_0, t_1)$ of $t_s$, $T = [t_0, t_1]$. By increasing $t$ we can achieve $\beta_{\text{CL}}(t) = -\beta_{\text{CL}}(t_s)$. To complete the proof for this case we set $\epsilon_t = -\beta_{\text{CL}}(t_s)$. Now consider the case where $\alpha$ is even and $k_{\alpha_t} < 0$. Then $\beta(t)$ has a local maximum when $t = t_s$. We can now choose an open neighborhood $(t_0, t_1)$ of $t_s$ such that $\beta(t)$ is increasing on $(t_0, t_1)$ and decreasing on $(t_1, t_2)$. By increasing $t$ we can achieve $\beta_{\text{CL}}(t) = \beta_{\text{CL}}(t_0)$. Once again we set $\epsilon_t = -\beta_{\text{CL}}(t_0)$ to establish the proof in this case. The same type of reasoning establishes the proof in the remaining cases. Shrinking $\epsilon_t$ if necessary we can also guarantee that $\beta_{\text{CL}}(t) \neq 0$, $\beta_{\text{CL}}(t) \neq 0$. □

Lemma 5.3: Suppose that $y_d : [t_0, t_1] \rightarrow \mathbb{R}^m$ satisfies the hypotheses of Theorem 4.2 and $x_d$ denotes the state trajectory which achieves exact tracking of $y_d$. Then there exists $\epsilon > 0$ such that $x_d$ intersects $M_{\text{CL}}$ transversally. That is, if $x_d(t) \in M_{\text{CL}}$ then $\beta(t) \not\in T_{x_d}(t)(M_{\text{CL}})$, the tangent space to $M_{\text{CL}}$ at $x_d(t)$.

Proof: First we note that $M_{\text{CL}}$ is a level set for the function $b(x) = \det(b(x))$, that is $M_{\text{CL}} = \{ x \in M \mid b(x) = 0 \}$. This means that for $x \in M$ the differential $db(x)$ is normal to the tangent space of $M_{\text{CL}}$ at $x$. In particular, if $x_d(t) \in M_{\text{CL}}$, then $\beta_{\text{CL}}(t) = \frac{\partial}{\partial t} \det(b(x_d(t))) = db(x_d(t)) \beta_{\text{CL}}(t)$. From Lemma 5.2 and the compactness of $[t_0, t_1]$ we can, for $\epsilon$ sufficiently small, ensure that the time $t$ has the properties of the times $t_a$ or $t_b$ identifies in Lemma 5.2. In particular, $\beta_{\text{CL}}(t) \neq 0$, which implies that $x_d(t) \not\in T_{x_d}(t)(M_{\text{CL}})$.

Proof of Theorem 5.1: Choose $\epsilon$ as in Lemma 5.3, let $G$ be a $m \times [p]$ matrix, and let $u_x(x, t)$ be the controller (21). To study the solutions to our state differential equation using this discontinuous controller, we construct a differential inclusion $\tilde{x} \in F^*(t, x)$ whose solutions satisfy the state differential equation almost everywhere (see Appendix A).

We begin by showing that the set-valued map $F^*$ satisfies the so-called basic conditions of [12], which implies that solutions exist. To simplify notation we define the vector fields $f^{\text{CL}}(x, t)$ and $f^{\text{OL}}(t)$ by

$$f^{\text{CL}}(x, t) = f(x) + g(x)u^\text{CL}_y(x, t)$$
$$f^{\text{OL}}(t) = f(x) + g(x)u^\text{OL}_y(t).$$

The set-valued map $F^*$ is then defined by

$$F^*(t, x) = \begin{cases} f^{\text{CL}}(x, t), & \det(b(x)) > \epsilon \\ f^{\text{OL}}(t), & \det(b(x)) \leq \epsilon \end{cases}$$

where $\text{conv}\{f^{\text{CL}}(x, t), f^{\text{OL}}(t)\}$ denotes the convex hull generated by $f^{\text{CL}}(x, t)$ and $f^{\text{OL}}(t)$. By construction, $F^*(t, x)$ is nonempty, compact, and convex. The proof that $F^* : [t_0, t_1] \times M \rightarrow \mathbb{R}^m$ is upper semicontinuous can be found in [12, pp. 67–68] and is omitted here. Theorem A.1 then guarantees that solutions to the differential inclusion $\dot{z}(t) \in F^*(t, x)$ exist.

There are several sufficient conditions which guarantee uniqueness of solutions [12, pp. 104–116]. We will now show that, for $x(t_0) = x_0$ sufficiently close to $x_0$, the differential inclusion $\dot{z}(t) \in F^*(t, x)$ has unique solutions. The simplest case occurs when $x(t_0) = x_0$. In this situation we can achieve exact tracking as a consequence of Theorem 4.2. In particular, $z \in F^*(t, x)$ has a unique solution $x_d$ which is continuously differentiable, the resulting output is $y = y_d$, and $u_x(x_d(t), t) = u^\text{CL}_y(x_d(t), t) = u^\text{CL}_y(x_d(t), t)$ also, from Lemma 5.3 we know that $x_d(t)$ intersects $M_{\text{CL}}$ transversally. That is, if $x_d(t_0) \in M_{\text{CL}}$ then $\dot{x}_d(t_0) \not\in T_{x_d(t_0)}(M_{\text{CL}})$, the tangent space to $M_{\text{CL}}$ at $x_d(t_0)$. This transversality can be
expressed in another way by noting that, on $M \setminus M_e$, we have $F^e(t, x) = \{f(x, t), \}$, where $f(x, t) = f(x) + g(x)u_e(x, t)$ is a smooth vector field. Now let $f^+$ and $f^-$ be the limiting values of $f(x_0(t), t)$ as $t$ approaches $t_0$ from the right and left, respectively. Then Lemma 5.3 implies that $f^+_N$ and $f^-_N$, the normal components of $f^+$ and $f^-$, are not equal to zero at $(x_d(t_0), t_0)$. Furthermore, since $x_d$ is smooth we have $f^+_N(x_d(t_0), t_0) = f^-_N(x_d(t_0), t_0) \neq 0$. Thus there is some open neighborhood $U_{t_0}$ of $x_d(t_0)$ in the separating manifold $M_e$ on which either $f^+_N > 0$, $f^-_N > 0$ or $f^+_N < 0$, $f^-_N < 0$. This allows us to use [12, Corollary 1, p. 107] to conclude that solutions to our differential inclusion $\dot{x} \in F^e(t, x)$ which pass through $U_{t_0}$ are unique. We know from Lemma 5.3 and the compactness of $[t_0, t_1]$ that $x_d$ enters the separating manifold $M_e$ at most a finite number of times. Thus, by keeping $x_d(t)$ sufficiently close to $x_0$ we can guarantee that the resulting trajectory is the unique solution to our differential inclusion.

From [12, pp. 87–88] it follows that if solutions are unique then they depend continuously on the initial conditions. This implies that there exists an open neighborhood $U_0$ of $x_0$ such that, for $x(t_0) = x_0 \in U_0$, the solution $x(t)$ to the differential inclusion $\dot{x} \in F^e(t, x)$ satisfies $|x(t) - x_0| < \delta$ for all $t \in T$. In particular, $|x(t) - x_0| < \delta$ for all $t \in [t_0, t_1]$. The boundedness of $u_e(x, t)$ follows from the compactness of $T$ and the fact that our state trajectory is piecewise smooth.

Remark 5.4: Note that when $y_d$ satisfies conditions 1) and 2) of Theorem 4.2 the controller (21) is continuous and achieves exact tracking. Theorem 5.1 asserts that if condition 1) of Theorem 4.2 fails to hold, then we can keep $y$ arbitrarily close to $y_d$ by keeping the initial state $x_d$ close to $x_0$. The proof is based on the continuity of solutions to the differential inclusion $\dot{x} \in F^e(t, x)$ with respect to the initial conditions. Thus to ensure that $y$ stays close to $y_d$ we might have to require that the actual initial state $x_d$ be extremely close to $x_0$. Furthermore, the tracking error could grow with time. We have not yet exploited our ability to affect the error dynamics through the choice of the matrix $G$ and constant $\epsilon$ which appear in the definition of the discontinuous controller (21).

To study and control the tracking error directly we introduce an ordinary differential equation in $H^1$, the space of outputs and their derivatives. Suppose that $x(t)$ and $y(t)$ are the state and output trajectories for (1) which result when the input $u(t)$ is applied. From (4) we have $y(t) = a(x(t)) + b(x(t))u(t)$.

If $u = u_0^{CL}$, then the tracking error $e_y$ satisfies

$$
eq a(x(t)) + b(x(t))[b(x(t))]^{-1} \times [y_d(t) - G' \dot{x}_y(t)] - y_d(t)$$

$$= G(E^y - \dot{x}_y).$$

Thus $\dot{e}_y(t)$ is the solution to a linear differential equation of the form

$$\dot{e}_y = G^E e_y$$

where $G^E$ is an $[p] \times [p]$ matrix. Given a set of desired eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$ for $G^E$, we can choose $G$ so that $G^E$ is in rational canonical form with eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$. In particular, we can ensure that the error converges to zero quickly. Unfortunately, the controller $u_d^{CL}$ becomes unbounded as the state approaches $M_e$, and so our controller $u_e$ switches to $u_d^{OL}$ when $x(t)$ enters $M_e$. Thus, while $x(t)$ is close to $M_e$, we will have

$$\dot{e}_y^{(s)} = y^{(s)} - y_d^{(s)} = a(x(t)) + b(x(t))u_d^{OL} - y_d^{(s)}$$

Thus the evolution of $e_y$ is governed by (24) for the length of time that the open-loop controller is used. This time interval, in turn, depends on the size of $\epsilon$. The ideal situation is when (23) holds for all time $t \in [t_0, t_1]$. We will denote this ideal tracking error by $e_0$ so that

$$\dot{e}_y^{(0)} = G^E(t - t_0) \dot{e}_y(t_0)$$

$$= G^E(t - t_0) [y_d(t_0) - y_d^{(0)}].$$

If the state trajectory reaches $M_e$, then $u_e = u_d^{OL}$ over some intervals of time, and we will not achieve the above reduction of the tracking error for all $t \in T$. We denote the actual output tracking error by $\dot{e}_y = y - y_d$. The next theorem shows that $\dot{e}_y$ converges uniformly to $e_0$ as $e_i \to 0$.

Theorem 5.5: Suppose that $y_d : [t_0, t_1] \to H^1$ satisfies the hypotheses of Theorem 4.2 and thus can be tracked exactly by the output of system (1), provided that $x(t_0) = x_0$. Then, there exists an open neighborhood $U_0$ of $x_0$ and a decreasing sequence $\{\epsilon_i > 0\}_{i=1}^\infty$ such that, for every $x_0' \in U_0$, the discontinuous controller $u_e$ produces a unique state trajectory and the output tracking error $e_y(t)$ satisfies

$$\|y(t) - y_d(t)\| \to 0$$

uniformly on $[t_0, t_1]$ as $i \to \infty$.

Proof: From Theorem 5.1 we know that there exists an open neighborhood $U_0$ of $x_0$ in $M$ and $\epsilon > 0$ such that $x_0' \in U_0$ the differential inclusion $\dot{x} \in F^e(t, x)$ has unique solutions which depend continuously on the initial conditions. Let $x_d$ denote the state trajectory with $x(t_0) = x_0$. It follows that $h^e(x_d(t)) = y_d(t)$ and exact tracking is achieved when $x(t_0) = x_0$. Now let $\epsilon_i$ be a positive constant with $0 < \epsilon_i < \epsilon$. The control $u_e$ still results in the state trajectory $x_d$ when $x(t_0) = x_0$. As $\epsilon_i$ goes to zero, the control $u_e$ approaches $u_d^{CL}$. This means that the solutions to our differential inclusion stay close to $x_d$ for $x(t)$, close to $x_0$, and $0 < \epsilon_i < \epsilon$. For a proof of the continuity of solutions to our differential inclusion with respect to $\epsilon$ (or the control $u_e$) see [12, Th. 3 p. 92]. Thus we can choose an open neighborhood $U_0 \subset U_0'$ of $x_0$ with the following property: for $x(t_0) \in U_0$, $0 < \epsilon_i < \epsilon$, and control $u = u_e$, the differential inclusion $\dot{x} \in F^e(t, x)$ produces a unique state trajectory. Of course $\epsilon_i$ approaches zero as the output tracking error $e_0$ approaches the desired error $\dot{e}_y$. It follows that for each $x_0' \in U_0$ we have $\lim_{i \to \infty} \|e_i - \epsilon_i\| = 0$, where the convergence is uniform with respect to $t$ as $T$ is compact.

Remark 5.6: If $y_d$ is defined on $[t_0, \infty)$ and has a finite set of singular times $T_s$, then the controller (21) can be used on $M \times [t_0, \infty)$ to achieve asymptotic output tracking of $y_d$. 


VI. A CASE STUDY: CHAIN SYSTEMS

Nonholonomic systems constitute a particularly interesting class of nonlinear control systems. For one thing, they are described by state equations without drift term; Brock-ett proved that there is no smooth state feedback which stabilizes such systems [4]. For another, the velocities of a nonholonomic system must satisfy a certain number of nonintegrable constraints; therefore—even for a completely controllable system—the motion planning problem proves to be difficult. Among nonholonomic systems, the so-called chain systems or systems in extended Goursat form are particularly amenable to analysis and yet they exhibit the same interesting properties of the general case. In this section we will consider the simplest case of a chain system, namely, the case of a system with three states and two controls. We shall show that, after suitable dynamic extension, this system satisfies Assumptions A1 and A2 and hence we can apply our results on exact and approximate output tracking. As a matter of fact, the analysis that we develop in this section could be applied to chain systems of higher dimension. For the sake of conciseness, we restrict our attention to the simplest case.

A. Dynamic Extension of a Chain System

Consider the chain system

\[
\begin{align*}
\dot{x}_1 &= \bar{u}_1 \\
\dot{x}_2 &= x_3 \bar{u}_1 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= u_1
\end{align*}
\]  

(27)

and define the output function

\[
y = h(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]  

(28)

Even though the choice of the output function (28) may appear arbitrary, the results that we present in this section remain valid for any other output function which is obtained from (28) by a proper generalized output transformation, in the sense of [1].

Straightforward computations show that the system (27) with output map (28) is invertible, in the sense of [9]–[11]. Thus, the dynamic extension \( \bar{u}_1 = x_4, \quad \dot{x}_4 = u_1 \), the system (27) and (28) has a well-defined vector relative degree, as defined in Section II. To save accounting, we consider the following extended system:

\[
\begin{align*}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_3 x_4 \\
\dot{x}_3 &= u_2 \\
\dot{x}_4 &= u_1
\end{align*}
\]  

(29)

B. Control of a Chain System: Exact Output Tracking

Taking successive derivatives of the output function along the trajectories of the system (29), we obtain

\[
\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = b(x)u = \begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

Therefore, Assumption A1 is satisfied; namely, the rank of the matrix \( b(x) \) is two everywhere but on the singular set, which in turn is defined by \( M_\varepsilon = \{ x \in \mathbb{R}^4 \mid x_4 = 0 \} \). On the other hand, the relative orders of the map \( h(x) \) are \( r_1 = r_2 = 2 \). Therefore \( |r| = 4 \) and the map \( h^r : M_\varepsilon \rightarrow \mathbb{R}^4 \) is given by \( x \mapsto (x_1, x_3, x_2, x_3 x_4) \). Next, note that \( N_\varepsilon = h^r(M_\varepsilon) = \{ (y_1, y_2, y_2 x_3) \mid \dot{y}_1 = 0 \} \). Therefore, \( N_\varepsilon = \{ (x_1^2, x_2, 2x_3, x_4) \in \mathbb{R}^4 \mid x_2 = 0 \} \). It follows that Assumption A2 also holds with \( A(\varepsilon) = 0 \) and

\[
B(\varepsilon) = \begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix}.
\]

(30)

To see this notice that \( a(x) = 0 \), and that, off \( N_\varepsilon \), we have

\[
b(x) = h^r(x) = \begin{pmatrix} 1 & 0 \\ x_3 & x_4 \end{pmatrix}.
\]

(31)

Following the notation of Section III we define \( \beta_\varepsilon(t) = \det B(h^r(t)) = \dot{y}_1(t) \) and

\[
\begin{pmatrix} \dot{F}_{y_1}(t) \\ \dot{F}_{y_2}(t) \end{pmatrix} = \begin{pmatrix} \dot{y}_1(t) \dot{y}_2(t) \\ \dot{y}_2(t) - \frac{\dot{y}_1(t) \dot{y}_2(t)}{\dot{y}_1(t)} \end{pmatrix}.
\]

Let \( t_\varepsilon \in [t_0, t_1] \) be such that \( \beta_\varepsilon(t_\varepsilon) = 0 \). Note that \( F_{y_1}(t_\varepsilon) = 0 \). Therefore, an output trajectory \( t \mapsto y(t) \) can be tracked by the output of the system (29) only if (we drop the \( t \) argument for convenience)

\[
\lim_{t \to t_\varepsilon} F_{y_2} = \lim_{t \to t_\varepsilon} \left[ y_2 - \frac{\dot{y}_1 \dot{y}_2}{\dot{y}_1} \right] = \dot{y}_2 - \lim_{t \to t_\varepsilon} \frac{\dot{y}_2}{\dot{y}_1} = 0.
\]

(32)

In particular, if \( \dot{y}_2 \to 0 \) as \( \dot{y}_1 \to 0 \) and \( \dot{y}_1(t_\varepsilon) \neq 0 \), then an application of de L’Hospital’s rule ensures that (30) holds. The fact that \( \beta_\varepsilon(t_\varepsilon) = \dot{y}_1(t_\varepsilon) \neq 0 \) means that the degree of singularity of the map \( t \mapsto y(t) \) at time \( t = t_\varepsilon \) is one. It is clear that \( \dot{F}_{y_1} = \dot{y}_1^2 + \dot{y}_1 \dot{y}_2 \) is well defined for every \( t \in [t_0, t_1] \). On the other hand, a lengthy but straightforward computation shows that the limit

\[
\lim_{t \to t_\varepsilon} \dot{F}_{y_2} = y_2^3 - \lim_{t \to t_\varepsilon} \frac{\dot{y}_2 \dot{y}_1 y_1 + y_2^3 \dot{y}_1 - \dot{y}_2^2 \dot{y}_1}{\dot{y}_1} \exists \text{ (this amounts to apply de L’Hospital’s rule twice to the second term of the last expression)}.
\]

Summing up, an output trajectory \( t \mapsto y(t) \) can be tracked by the output of the system (29) if the following condition holds:

\[
\dot{y}_1(t_\varepsilon) = 0 \Rightarrow \dot{y}_2(t_\varepsilon) = 0 \quad \dot{y}_1(t_\varepsilon) \neq 0.
\]

(33)

There are different ways to generate output maps which satisfy conditions (33). Typically, the method consists of proposing a trigonometric, polynomial, or spline function with sufficiently many degrees of freedom so as to fulfill (33) as well as some other control objectives. For instance, consider the family of output trajectories

\[
y_{da} = A \sin pt, \quad y_{d2} = B \cos qt.
\]

(34)

It is easily checked that the conditions in (33) are satisfied if and only if \( q/p \) is an even integer.
The open-loop control which generates the output trajectory (32) is given by

\[
 u_{OL} = \begin{cases} 
 \left( \frac{\dot{y}_{d1}(t)}{\dot{y}_{d1}(t)} \right), & \dot{y}_{d1}(t) \neq 0 \\
 \lim_{t \to t_s} \left( \frac{\dot{y}_{d1}(t)}{\dot{y}_{d1}(t)} \right), & \dot{y}_{d1}(t_s) = 0.
\end{cases}
\]  

(33)

In light of (31), and applying de L'Hospital's rule twice, we have

\[
u_{OL}(t) = \begin{cases} 
 \left( \frac{\ddot{y}_{d1}(t)}{\ddot{y}_{d1}(t)} \right), & \ddot{y}_{d1}(t) \neq 0 \\
 \left( \frac{\dddot{y}_{d1}(t_s)\dddot{y}_{d1}(t_s) - \dddot{y}_{d1}(t_s)\dddot{y}_{d1}(t_s)}{2\dot{y}_{d1}(t_s)} \right), & \dddot{y}_{d1}(t_s) = 0.
\end{cases}
\]  

(34)
As pointed out in Section IV, the open-loop controller (33) will generate the output (32) only if the initial state $x_0$ satisfies $h^T(x_0) = (y_{d1}(0), \dot{y}_{d1}(0), y_{d2}, \dot{y}_{d2}(0))$. If this condition is satisfied, then the controller (33) may be replaced by the closed-loop control (7). However, if $y(t)$ differs slightly from $y_d(t)$ due to initial errors or external perturbations, $y$ will diverge from $y_d$. In the next subsection we compute our discontinuous time-varying controller which avoids this problem.

C. Control of a Chain System: Approximate Output Tracking

Suppose that our control objective is that the output of (29) tracks the map (32). We can apply our result on approximate output tracking, Theorem 5.5, to the extended system (29). In particular we can use the discontinuous time-varying controller (21). Define

$$
\begin{align*}
\dot{e}_y &= \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} y_1 - y_{d1} \\ x_2 - y_{d2} \end{pmatrix} = \begin{pmatrix} x_1 - y_{d1} \\ x_2 - y_{d2} \end{pmatrix}, \\
\dot{e}_y &= \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 - \dot{y}_{d1} \\ \dot{y}_2 - \dot{y}_{d2} \end{pmatrix} = \begin{pmatrix} x_4 - \dot{y}_{d1} \\ x_3 x_4 - \dot{y}_{d2} \end{pmatrix},
\end{align*}
$$

With this notation, the controller (21) becomes

$$
\begin{align*}
u_{CL}(x, t) &= \begin{pmatrix} u_{CL1} \\ u_{CL2} \end{pmatrix}, \\
&= \begin{pmatrix} x_4 > \epsilon \\ x_4 \leq \epsilon \end{pmatrix} \quad \text{(35)}
\end{align*}
$$

where $u_{OL}$ is given by (34) and $u_{CL}$ is given by

$$
\begin{align*}
u_{OL}(x, t) &= \begin{pmatrix} u_{OL1} \\ u_{OL2} \end{pmatrix} = \begin{pmatrix} \hat{y}_{d1} - \hat{y}_{d2} - x_4 (x_3 y_{d1} - x_1 y_{d2}) \\ x_4 \end{pmatrix}, \\
&= \begin{pmatrix} \hat{y}_{d1} - \hat{y}_{d2} - x_4 (x_3 y_{d1} - x_1 y_{d2}) \\ x_4 \end{pmatrix}. \quad \text{(36)}
\end{align*}
$$

Fig. 1 shows the numerical results obtained with the following parameter values: $A = 1$, $B = 0.25$, $p = 1$, $q = 4$, $g_{d1} = g_{d2} = 3$, $g_{d1} = g_{d2} = 2$, and $\epsilon = 0.2$. The initial condition $x_0$ which satisfies $h^T(x_0) = (y_{d1}(0), \dot{y}_{d1}(0), y_{d2}(0), \dot{y}_{d2}(0))$ is $x_0 = (0, 0.25, 0, 1)$. Instead of $x_0$, we used the initial condition $x_0' = (0.15, 0.35, 0, 1)$. This represents the more realistic situation when the initial condition of the system is unknown. It is worth noting that the first time that the trajectories of the system cross the singular set, the error between the closed-loop control from the ideal open-loop control. However, after three units of time this error is so small that the closed-loop and the open-loop control become indistinguishable.

The tracking error can be reduced more rapidly by changing $G$. Implicit in the statement of Theorem 5.1 is that there are limits to how far one can deviate from $x_0$. Thus choosing $x(0)$ to be negative, for example, will result in sliding of the state trajectory along the manifold $M_\epsilon$ and there is no guarantee that the state will ever leave this submanifold. In this case, solutions to our differential inclusion need not be unique, and the tracking error will not be reduced.

D. Comparison with Bloch and Drakunov’s Approach

In [2] the so-called nonholonomic integrator is considered and a discontinuous time-varying controller is proposed, which achieves global approximate tracking of a given trajectory. The system considered in [2] is described by the equations $\dot{x} = u_1$, $\dot{y} = v$, $\dot{z} = xv - yu$, and after the change of coordinates $x_1 = x$, $x_2 = xy - z$, $x_3 = 2v$, $u = u_1$, $2v = u_2$, it takes precisely the form (27). Therefore, a modified version of the controller (34)–(36) could be used to achieve global approximate tracking for the nonholonomic integrator.

Both Bloch and Drakunov’s controller and ours are discontinuous and time-varying (they ought to be, since the nominal trajectory depends on time). The main difference between them is that, while the controller proposed in [2] produces sliding along the surface $z = z^* = 0$ (here $z^*$ is the desired $z$-trajectory), our controller avoids sliding along the switching manifolds $M_\epsilon = \{x \in M \mid \det(b(x)) = \epsilon\}$. This can be explained as follows: in Bloch and Drakunov’s approach, there is no guarantee that the nominal trajectory $X^* = (x^*, y^*, z^*)$ can be tracked by any continuous control law $u : [t_0, t_1) \rightarrow \mathbb{R}^2$. Therefore, the discontinuous controller proposed in [2] steers in finite time the state variables to some integral manifold where asymptotic tracking of the nominal trajectory is possible. On the other hand, we attempt to achieve approximate tracking of a restricted class of output trajectories, for which the exact problem can be solved by a continuous control law, provided that the initial condition is close to the ideal one. Hence, avoiding sliding along the switching surface is instrumental for our approach. Otherwise, it is not possible to ensure continuous dependence on the initial conditions.

Technically, the fact that there is no sliding along the switching manifold $M_\epsilon$ is ensured by Lemma 5.3 (see [12] for further details).

VII. CONCLUSIONS AND FINAL REMARKS

We have generalized the results of [16] on exact output tracking for SISO systems by weakening the observability assumptions and classifying the singularities which can appear in the MIMO nonlinear inversion algorithm. This let us identify a class of smooth output functions $y_d(t)$ which can be tracked exactly using continuous controls.

If $y(t)$ differs slightly from $y_d(t)$ due to initial errors or external perturbations, the exact tracking feedback controller may become unbounded. Our main result was a discontinuous time-varying controller which avoids this problem and achieves approximate output tracking of $y_d(t)$. By starting off with an output sufficiently close to $y_d$, our discontinuous controller produced a unique state trajectory and output $y$ and enabled us to regulate the tracking error $y - y_d$. In the case where $y_d$ is defined on $[t_0, \infty)$ and has a finite set of singular times $T_k$, the controller (21) achieves asymptotic output tracking of $y_d$ in the end.

Our results on approximate output tracking using discontinuous controls were applied to the problem of motion planning for a nonholonomic chain system.

APPENDIX

DIFFERENTIAL INCLUSIONS

We recall some basic facts about the theory of differential inclusions. For further details, the reader is referred to [6] and [12].
Let $G$ be an open and connected subset of $\mathbb{R}^n$ and consider the differential inclusion $\dot{x} \in F(t,x)$, where $(t,x) \in G$ and $F$ is a set-valued map. A solution of the differential inclusion is an absolutely continuous function $x : [t_0, t_1] \rightarrow G$ such that $\dot{x}(t) \in F(t, x(t))$ for almost all $t \in [t_0, t_1]$. We say that the set-valued map $F$ satisfies the basic conditions if $F$ is upper semicontinuous and for all $(t,x) \in G$, $F(t,x)$ is nonempty, compact, and convex.

**Theorem A.1** ([12, pp. 77, 78]): Let $F(t,x)$ satisfy the basic conditions on a compact set $D \subset G$. Then for any point $(t_0, x_0) \in D$ there is a solution to the differential inclusion $\dot{x} \in F(t,x)$ such that $x(t_0) = x_0$. Moreover, each solution can be continued on both sides up to the boundary of the compact set $D$.

**Theorem A.2** ([12, pp. 87, 88]): Let $F(t,x)$ satisfy the basic conditions on a compact set $D \subset G$ and suppose that for $t \geq t_0$ the differential inclusion $\dot{x} \in F(t,x)$ has a unique solution and let its graph on the segment $[t_0, t_1]$ lie within $G$. Let $x(t)$ and $x^*(t)$ denote solutions of the differential inclusion $\dot{x} \in F(t,x)$ satisfying, respectively, $x(t_0) = x_0$ and $x^*(t_0) = x_0$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x(x(t) - x(t))\| \leq \epsilon$ whenever $\|x_0 - x_0\| \leq \delta$, $\|t_0 - t_0\| \leq \delta$.

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