CONVERGENCE OF THE OPTIMAL VALUES OF CONSTRAINED MARKOV CONTROL PROCESSES*

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Abstract

We consider a sequence of discounted cost, constrained Markov control processes (CCPs) with countable state space, metric action set and possibly unbounded cost functions. We give conditions under which the sequence of optimal values of the CCPs converges to the optimal value of a limiting CCP, and, furthermore, the accumulation points of sequences of optimal policies for the CCPs are optimal policies for the limiting CCP. These results are obtained via an approximation theorem for general minimization problems.

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1 Introduction

In this paper we consider a sequence of constrained Markov control processes (hereafter abbreviated CCPs) that "converge" in some suitable sense to a limiting CCP, denoted CCP. The state space for each of these CCPs is a countable set, the control (or action) set is a metric space, the cost functions can be unbounded, and the criterion to be minimized is the infinite-horizon expected discounted cost. We give conditions under which the sequence of optimal values of the CCPs converges to the optimal value of CCP, and, furthermore, the accumulation points of sequences of optimal policies for the approximating CCPs are optimal policies for CCP. To obtain these results we first study the convergence of optimal values and solutions for general minimization problems, which is an important issue in itself. The corresponding convergence results are applied to CCPs by transforming them into minimization problems on sets of (occupation) measures. Hence, otherwise put, this paper studies the sensitivity of optimal values and solutions for general optimization problems and CCPs with respect to changes in some of their components.

Constrained control processes (CCPs) form an important class of stochastic control problems with applications in many areas (see for instance [1, 2, 5, 8-11, 16-18, 20-22, 24-27]), and they have been studied using different techniques, for instance, infinite-dimensional linear programming, dynamic programming, convex programming, and Pareto optimality. In particular, problems on the convergence of optimal values and optimal policies for CCPs were studied in [1, 2, 26] in different settings. Indeed, in [1, 26] conditions were obtained for convergence in the discount factor, and [1] also considered the convergence in the planning horizon and of truncations in the state space. Our work differs of [1, 26] in the assumptions and techniques, because we consider the joint convergence in the transition law, the cost-per-stage, the initial distribution, and the constraints. Similar convergence results to those considered in this paper were studied in [2] for CCPs with finite state space and finite action set. In fact, our approach here generalizes [2] to CCPs with countable state spaces and metric action spaces.

On the other hand, our results on convergence of values for general optimization problems, generalizes in particular the results in [6, 23] (see Remark 2.4).

The remainder of the paper is organized as follows. In section 2 we study the convergence of general optimization problems (see Theorem 2.3). In section 3 we introduce the CCPs we are concerned with and state our main result on convergence of values for CCPs. The results in section 3 are proved in section 5, after introducing important results in section 4 concerning the so-
called occupation measures and their relation with stationary policies. We conclude in section 6 with some general comments on alternative assumptions for the convergence results in section 3.

2 General minimization problems

We shall use the notation $\mathbb{N} := \{1, 2, \ldots\}$, $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$.

Let $X$ be a topological Hausdorff space. For each $n \in \overline{\mathbb{N}}$, consider a function $\varphi_n : X \to \overline{\mathbb{R}}$, a set $\mathcal{F}_n \subset X$, and the minimization problem

$$\mathbf{P}_n : \text{Minimize } \varphi_n(x)$$

subject to: $x \in \mathcal{F}_n$.

We call $\mathcal{F}_n$ the set of feasible solutions for $\mathbf{P}_n$. If $\mathcal{F}_n$ is nonempty, the (optimum) value of $\mathbf{P}_n$ is defined as $\inf \mathbf{P}_n := \inf \{\varphi_n(x) \mid x \in \mathcal{F}_n\}$; otherwise, $\inf \mathbf{P}_n := +\infty$. The problem $\mathbf{P}_n$ is said to be solvable if there is a feasible solution $x^*$ that achieves the optimum value. In this case, $x^*$ is called an optimal solution for $\mathbf{P}_n$, and the value $\inf \mathbf{P}_n$ is then written as $\min \mathbf{P}_n = \varphi_n(x^*)$. We shall denote by $\mathcal{M}_n$ the minimum set, that is, the set of optimal solutions for $\mathbf{P}_n$.

To state our assumptions we will use Kuratowski's [19] concept of outer and inner limits of $\{\mathcal{F}_n\}$, denoted by $\mathcal{O}L\{\mathcal{F}_n\}$ and $\mathcal{I}L\{\mathcal{F}_n\}$, and defined as follows.

$$\mathcal{O}L\{\mathcal{F}_n\} := \{x \in X \mid x = \lim_{n \to \infty} x_{n_i}, \text{ such that } x_{n_i} \in \mathcal{F}_{n_i} \text{ for all } i\}.$$ 

Thus a point $x \in X$ is in $\mathcal{O}L\{\mathcal{F}_n\}$ if $x$ is an accumulation point of a sequence $\{x_n\}$ with $x_n \in \mathcal{F}_n$ for all $n$. On the other hand, if $x$ is the limit of the sequence $\{x_n\}$ itself, then $x$ is in the inner limit $\mathcal{I}L\{\mathcal{F}_n\}$, i.e.,

$$\mathcal{I}L\{\mathcal{F}_n\} := \{x \in X \mid x = \lim_{n \to \infty} x_n, \text{ where } x_n \in \mathcal{F}_n \text{ for all but a finite number of } n\}'s\}.$$

In these definitions we may, of course, replace $\{\mathcal{F}_n\}$ with any other sequence of subsets of $X$. Also note that $\mathcal{I}L\{\cdot\} \subset \mathcal{O}L\{\cdot\}$.

We shall consider two sets of hypotheses.

**Assumption 2.1** (a) The minimum sets $\mathcal{M}_n$ satisfy that

$$\mathcal{O}L\{\mathcal{M}_n\} \subset \mathcal{F}_\infty. \quad (2.1)$$
(b) If $x_{n_i}$ is in $\mathcal{M}_{n_i}$ for all $i$ and $x_{n_i} \to x$ (so that $x$ is in $\text{OL}(\mathcal{M}_{n})$), then

$$\liminf_{i \to \infty} \varphi_{n_i}(x_{n_i}) \geq \varphi_{\infty}(x).$$

(2.2)

(c) For each $x \in \mathcal{F}_{\infty}$ there exists $N \in \mathbb{N}$ and a sequence $\{x_n\}$ with $x_n \in \mathcal{F}_n$ for all $n \geq N$, and such that $x_n \to x$ and $\lim_{n \to \infty} \varphi_n(x_n) = \varphi_{\infty}(x)$.

**Assumption 2.2** Parts (b) and (c) are the same as in Assumption 2.1. Moreover:

(a) The minimum sets $\mathcal{M}_n$ satisfy that

$$\text{IL}(\mathcal{M}_n) \subset \mathcal{F}_{\infty}.$$  

(2.3)

Note that Assumption 2.1(c) implies, in particular, that $\mathcal{F}_{\infty} \subset \text{IL}(\mathcal{F}_n)$. This fact together with (2.3) yields that

$$\text{IL}(\mathcal{M}_n) \subset \mathcal{F}_{\infty} \subset \text{IL}(\mathcal{F}_n).$$

**Theorem 2.3** (a) If Assumption 2.1 holds, then

$$\text{OL}(\mathcal{M}_n) \subset \mathcal{M}_{\infty},$$

(2.4)
i.e., if $x_{n_i} \in \mathcal{M}_{n_i}$ converges to $x$, then $x$ is optimal for $\mathcal{P}_{\infty}$; furthermore, the optimal values of $\mathcal{P}_{n_i}$ converge to the optimal value of $\mathcal{P}_{\infty}$, that is,

$$\min \mathcal{P}_{n_i} = \varphi_{n_i}(x_{n_i}) \to \varphi_{\infty}(x) = \min \mathcal{P}_{\infty}.$$  

(2.5)

(b) Suppose that Assumption 2.2 holds. Then

$$\text{IL}(\mathcal{M}_n) \subset \mathcal{M}_{\infty}.$$

If in addition $\text{IL}(\mathcal{M}_n)$ is nonempty, then

$$\min \mathcal{P}_n \to \min \mathcal{P}_{\infty}.$$

**Proof.** We only prove (a) because the proof of (b) is quite similar.

To prove (a), let $x \in X$ be in the outer limit $\text{OL}(\mathcal{M}_n)$. Then there is a sequence $\{n_i\} \subset \mathbb{N}$ and $x_{n_i} \in \mathcal{M}_{n_i}$ for all $i$ such that

$$x_{n_i} \to x.$$  

(2.6)

Moreover, by Assumption 2.1(a), $x$ is in $\mathcal{F}_{\infty}$. To prove that $x$ is in $\mathcal{M}_{\infty}$, choose an arbitrary $x' \in \mathcal{F}_{\infty}$ and let $\{x'_n\}$ and $N$ be as in Assumption 2.1(c) for $x'$, that
is, \( x'_n \) is in \( \mathcal{F}_n \) for all \( n \geq N \), \( x'_n \to x' \), and \( \varphi_n(x'_n) \to \varphi_\infty(x') \). Furthermore, if \( \{n_i\} \subseteq \mathbb{N} \) is as in (2.6), then the subsequence \( \{x'_{n_i}\} \) of \( \{x'_n\} \) also satisfies
\[
x'_{n_i} \text{ is in } \mathcal{F}_{n_i}, \quad x'_{n_i} \to x', \quad \text{and} \quad \varphi_{n_i}(x'_{n_i}) \to \varphi_\infty(x'). \tag{2.7}
\]
Combining the latter fact with Assumption 2.1(b) and the optimality of \( x_{n_i} \) we get
\[
\varphi_\infty(x) \leq \liminf_{i \to \infty} \varphi_{n_i}(x_{n_i}) \quad \text{(by (2.2))}
\leq \liminf_{i \to \infty} \varphi_{n_i}(x'_{n_i})
= \varphi_\infty(x') \quad \text{(by (2.7)).}
\]
Hence, as \( x' \in \mathcal{F}_\infty \) was arbitrary, it follows that \( x \) is in \( M_0 \), that is, (2.4) holds.

To prove (2.5), suppose again that \( x \) is in \( \text{OL}\{M_n\} \) and let \( x_{n_i} \in M_{n_i} \) be as in (2.6). By Assumption 2.1(c), there exists a sequence \( x'_{n_i} \in \mathcal{F}_{n_i} \) that satisfies (2.7) for \( x \) instead of \( x' \); thus
\[
\varphi_\infty(x) \leq \liminf_{i \to \infty} \varphi_{n_i}(x_{n_i}) \quad \text{(by (2.2))}
\leq \limsup_{i \to \infty} \varphi_{n_i}(x_{n_i})
\leq \limsup_{i \to \infty} \varphi_{n_i}(x'_{n_i})
= \varphi_\infty(x) \quad \text{(by (2.7)).}
\]
This proves (2.5).

\[\square\]

**Remark 2.4** Parts (a) and (b) of Theorem 2.3 generalize in particular the results in [23] and [6], respectively. Indeed, using our notation, in [6, 23] it is assumed that the cost functions \( \varphi_n \) converge uniformly to \( \varphi_\infty \). On the other hand, with respect to the feasible sets \( \mathcal{F}_n \) in [6] it is assumed that \( \text{OL}\{\mathcal{F}_n\} = \mathcal{F}_\infty \), whereas in [23] it is required that \( \mathcal{F}_n \to \mathcal{F}_\infty \) in the Hausdorff metric. These hypotheses trivially yield the following conditions:

(i) The inner and/or the outer limit of the feasible sets \( \mathcal{F}_n \) coincide with \( \mathcal{F}_\infty \), i.e.
\[
\text{OL}\{\mathcal{F}_n\} = \text{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty \tag{2.8}
\]
or
\[
\text{IL}\{\mathcal{F}_n\} = \mathcal{F}_\infty. \tag{2.9}
\]
(ii) For each sequence \( \{x_n\} \subset \mathcal{X} \) such that \( x_n \to x \) it holds that
\[
\lim_{n \to \infty} \varphi_n(x_n) = \varphi_\infty(x). \tag{2.10}
\]

However, instead of (2.8) and (2.10) we require (the weaker) Assumption 2.1, and instead (2.9) and (2.10) we require (the weaker) Assumption 2.2.
3 Convergence of values for CCPs

In this section we first introduce the constrained control problem we are interested with (see (3.1)-(3.3)), which is then reformulated (in (3.11)-(3.12)) using occupation measures. We next state our main convergence results, Theorem 3.9 and Corollary 3.10, and conclude the section with an example on a controlled queueing system.

Constrained Markov control models. The material in this introductory part is quite standard (see for instance [1, 5, 8-11, 16-18, 21-22, 24-26]).

For each $n \in \mathbb{N}$, the constrained Markov control model is of the form

$$(X, A, \{A(x) \mid x \in X\}, Q_n, c_n^0, \gamma_n, e_n, k_n).$$

The spaces $X$ and $A$ are the state space and the control (or action) set, respectively. We shall assume that $X$ is a countable set (with the discrete topology), and $A$ is a metric space endowed with the corresponding Borel $\sigma$-algebra $B(A)$. For each state $x \in X$, the nonempty set $A(x) \in B(A)$ in (3.1) stands for the set of feasible control actions in $x$, and we shall suppose that it is compact. Note that the set

$$\mathbb{I} := \{(x, a) \mid x \in X, a \in A(x)\}$$

of feasible state-action pairs is a closed (hence a Borel-measurable) subset of $X \times A$. Moreover, $Q_n$ stands for the transition law, which is a stochastic kernel on $X$ given $\mathbb{I}$; $c_n^0 : \mathbb{I} \rightarrow \mathbb{R}$ is a measurable function that denotes the cost-per-stage; and $\gamma_n$ is the initial distribution, a probability measure (p.m.) on $X$. Thus, the first six components of (3.1) correspond to the standard unconstrained Markov control model

$$(X, A, \{A(x) \mid x \in X\}, Q_n, c_n^0, \gamma_n)$$

for each $n \in \mathbb{N}$. The last two components in (3.1), on the other hand, are a measurable function $c_n = (c_n^0, \ldots, c_n^0) : \mathbb{I} \rightarrow \mathbb{R}^q$ and a vector $k_n = (k_n^1, \ldots, k_n^q)$, which are used to define the constrained problem CCP$_n$ in (3.3), below.

To state the problem CCP$_n$ let us first recall that a control policy is a sequence $\pi = \{\pi_t\}$ of stochastic kernels $\pi_t$ that satisfy the constraint $\pi_t(A_{x_t}) = 1$ for every "history" $h_t = (x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t)$ and $t = 0, 1, \ldots$, where $(x_i, a_i)$ is in $\mathbb{I}$ for $i = 0, 1, \ldots, t - 1$. The set of all control policies is denoted by $\Pi$.

Let $\Phi$ be the family of stochastic kernels $\varphi$ on $A$ given $X$ such that $\varphi(A(x) \mid x) = 1$ for all $x \in X$. 

A control policy \( \pi = \{\pi_t\} \) is said to be randomized stationary (or simply stationary) if there exists a stochastic kernel \( \varphi \in \Phi \) such that \( \pi_t(\cdot|h_t) = \varphi(\cdot|x_t) \) for each history \( h_t \) and \( t = 0, 1, \ldots \). The family of randomized stationary policies will be identified with the set \( \Phi \).

Let \( \mathcal{P}(X) \) be the class of p.m.'s on \( \mathcal{B}(X) \). For each policy \( \pi \in \Pi \) and each \( n \in \mathbb{N} \), there exists a p.m. \( P_{\gamma_n}^{\pi} \) and a stochastic process \( \{(x_t, a_t), t = 0, 1, \ldots\} \) defined on a canonical measurable space \( (\Omega, \mathcal{F}) \), where \( x_t \) and \( a_t \) represent the state and the control variables at time \( t \). The expectation operator with respect to \( P_{\gamma_n}^{\pi} \) is denoted by \( E_{\gamma_n}^{\pi} \). If \( \gamma_n \) is concentrated at the initial state \( x_0 = x \), then we write \( P_{\gamma_n}^{\pi} \) and \( E_{\gamma_n}^{\pi} \) as \( P_x^{\pi} \) and \( E_x^{\pi} \), respectively.

**Constrained problem \( \text{CCP}_n \).** Let \( \alpha \in (0, 1) \) be a discount factor, and \( n \in \mathbb{N} \). For each policy \( \pi \in \Pi \), consider the \( \alpha \)-discounted cost function

\[
V_n^i(\pi) := (1 - \alpha) E_{\gamma_n}^{\pi} \left[ \sum_{t=0}^{\infty} \alpha^t c_n^i(x_t, a_t) \right] \quad \text{for } i = 0, \ldots, q. \tag{3.2}
\]

With this notation, for each \( n \in \mathbb{N} \) we may then define the constrained control problem \( \text{CCP}_n \), we are concerned with as follows:

\[
\text{CCP}_n : \quad \text{Minimize } V_n^0(\pi) \tag{3.3}
\]

subject to: \( V_n^i(\pi) \leq k_n^i \) (\( i = 1, \ldots, q \)), \( \pi \in \Pi \).

When \( c_n(x, a) = k \), for all \( (x, a) \in \mathcal{K} \), the constrained control problem \( \text{CCP}_n \) becomes the usual unconstrained control problem.

We say that \( \text{CCP}_n \) is consistent if the set

\[
\Delta_n := \{ \pi \in \Pi \mid V_n^0(\pi) < \infty, \quad \text{and} \quad V_n^i(\pi) \leq k_n^i \ \text{for} \ i = 1, 2, \ldots, q \}
\]

of feasible policies for \( \text{CCP}_n \) is nonempty. If \( \text{CCP}_n \) is consistent and \( \pi^* \in \Delta_n \) minimizes \( V_n^0(\pi) \) over all \( \pi \in \Delta_n \), then \( \pi^* \) is said to be an optimal policy for \( \text{CCP}_n \).

We shall introduce assumptions that, in particular, explain the sense in which \( \text{CCP}_n \) "converges" to \( \text{CCP}_\infty \). We shall distinguish two cases for the cost functions \( c_n^i \), the bounded case and the unbounded case. (Assumptions 3.1(c), on the law transition, and 3.2, on the costs, can be replaced with other conditions; see Remark 6.1)

**Assumption 3.1** For each \( n \in \mathbb{N} \),

(a) the set \( \Delta_n \) is nonempty, i.e. \( \text{CCP}_n \) is consistent, and
(b) the transition law \( Q_n(y|x, \cdot) \) is continuous on \( A(x) \) for each \( y, x \) in \( X \)

On the other hand, as \( n \to \infty \),

(c) \( Q_n(y|x, a) \to Q_\infty(y|x, a) \) for each \( x, y \in X \) and \( a \in A(x) \), and the convergence is uniform on \( A(x) \),

(d) \( \gamma_n(x) \to \gamma_\infty(x) \) for all \( x \in X \) (see Remark 3.4), and

(e) \( k_n \to k_\infty \).

We next introduce the hypotheses on the cost functions.

**Assumption 3.2 (Bounded case)**

(a) For each \( i = 0, 1, \ldots, q \) and \( n \in \mathbb{N} \),
the cost \( c_n^i \) is bounded and lower semicontinuous (l.s.c.) on \( I^i K \).

(b) \( c_n^i \to c_\infty^i \) uniformly on \( K \) for each \( i = 0, 1, \ldots, q \).

**Assumption 3.3 (Unbounded case)**

(a) The cost \( c^i_n(x, \cdot) \) is continuous on \( A(x) \) for each \( x \in X, i = 0, 1, \ldots, q \), and \( n \in \mathbb{N} \).

(b) There exist a finite measure \( \nu_0 \) on \( X \) and a nonnegative function \( g : X \to \mathbb{R} \) such that

\[
\sum_{x \in X} g(x) \nu_0(x) < \infty, \tag{3.4}
\]

\[
Q_n(\cdot|x, a) \leq \nu_0(\cdot) \ \forall \ (x, a) \in K, \ n \in \mathbb{N}, \tag{3.5}
\]

\[
|c^i_n(x, a)| \leq g(x) \ \forall \ (x, a) \in K, \ n \in \mathbb{N} \ i = 0, 1, \ldots, q. \tag{3.6}
\]

(c) For each \( i = 0, 1, \ldots, q \), \( c^i_n(x, a) \to c^i_\infty(x, a) \) for all \( (x, a) \in K \), and the convergence is uniform on \( A(x) \).

**Remark 3.4**

(a) Let \( S \) be a metric space (with the Borel-algebra). Let \( \mathbb{P}(S) \) the set of p.m.'s on \( S \), and \( C_b(S) \) the space of continuous bounded functions on \( S \). A sequence \( \{\nu_n\} \) in \( \mathbb{P}(S) \) is said to converge weakly to \( \nu \in \mathbb{P}(S) \) if

\[
\int_S u \, d\nu_n \to \int_S u \, d\nu \ \forall \ u \in C_b(S). \tag{3.7}
\]

If \( S \) is a Borel space, then so is \( \mathbb{P}(S) \) endowed with the topology of weak convergence (3.7). On the other hand, as the space \( X \) in (3.1) is a countable set, for a sequence \( \{\nu_n\} \) of p.m.'s on \( X \) the definition (3.7) of weak convergence becomes

\[
\sum_{x \in X} u(x) \nu_n(x) \to \sum_{x \in X} u(x) \nu(x)
\]
for every bounded function \( u \) on \( X \), which in turn is equivalent to
\[
\nu_n(x) \to \nu(x) \quad \forall \ x \in X.
\] (3.8)

Thus another way of stating Assumption 3.1(c) is \( \gamma_n \to \gamma_{\infty} \) weakly. Finally, note that, by Scheffé’s Theorem (see e.g. p. 233 in [4]), (3.8) is equivalent to
\[
\sum_{x \in X} |\nu_n(x) - \nu(x)| \to 0.
\] (3.9)

(b) (Weak topology on \( \Phi \).) A sequence \( \{\varphi_n\} \) in \( \Phi \) is said to converge weakly to \( \varphi \in \Phi \) if the sequence \( \{\varphi_n(\cdot | x)\} \) in \( P(A(x)) \) converges weakly to \( \varphi(\cdot | x) \) for each \( x \in X \).

(c) Let \( S \) and \( P(S) \) be as in (a). A subset \( P \) of \( P(S) \) is said to be relatively compact if for every sequence \( \{\nu_n\} \) in \( P \) there exists a subsequence \( \{\nu_{n_k}\} \) and a p.m. \( \nu \) in \( P(S) \) (but not necessarily in \( P \)) such that \( \nu_{n_k} \to \nu \) weakly. In this case, \( \nu \) is called a weak accumulation point of \( P \). If in addition \( P \) contains all of its weak accumulation points, then \( P \) is sequentially compact.

Let \( M(K) \) be the vector space of finite signed measures \( \mu \) on \( X \times A \), concentrated on \( K \) (i.e. \( \mu(K^c) = 0 \), where \( K^c \) denotes the complement of \( K \)), and let \( M^+(K) \) be the cone of nonnegative measures in \( M(K) \). We denote by \( P(K) \) the set of p.m.’s in \( M^+(K) \). If \( \mu \) is in \( M(K) \), we denote by \( \mu(X) \) its marginal (or projection) on \( X \), that is, \( \mu(X \times A) \) for each \( x \in X \). Moreover, sometimes we write an integral \( \int u \, d\mu \) as \( \langle \mu, u \rangle \) for any function \( u \) on \( K \) for which the integral is well defined.

It is well known that the constrained control problems \( CCP_n \) can be posed as linear programs using the occupation measures defined as follows.

**Definition 3.5 (Occupation measures)** A measure \( \mu \in M^+(K) \) is said to be an occupation measure for \( CCP_n \) if
\[
\widehat{\mu}(x) = (1 - \alpha)\gamma_n(x) + \alpha \int_K Q_n(x|y,a)\mu(d(y,a)) \quad \forall \ x \in X.
\] (3.10)

As \( \widehat{\mu}(X) = \mu(X \times A) \), it follows from (3.10) that an occupation measure and its marginal are p.m.’s. The occupation measures and the cost functions in (3.2) are related by the following proposition (for a proof see [12], p. 140, for instance).

**Proposition 3.6** (a) For each \( \pi \in \Pi \) there exists an occupation measure \( \mu \in M^+(K) \) such that
\[
V_n^i(\pi) = \langle \mu, c_n^i \rangle \quad \forall \ i = 0, 1, \ldots, q.
\]
(b) Conversely, if $\mu \in M^+(IK)$ is an occupation measure, then there exists an stationary policy $\varphi \in \Phi$ such that

\[ V^i_n(\varphi) = \langle \mu, c^i_n \rangle \quad \forall \; i = 0, 1, \ldots, q. \]

This proposition implies in particular that the search of optimal policies can be restricted to the class $\Phi$ of stationary policies. More explicitly, we have:

**Corollary 3.7** For each policy $\pi \in \Pi$, there exists a stationary policy $\varphi \in \Phi$ such that $V^i_n(\pi) = V^i_n(\varphi)$ for all $i = 0, 1, \ldots, q$. Therefore, if there exists an optimal policy $\pi \in \Pi$ for $CCP_n$, then it also exists a stationary policy $\varphi \in \Phi$ that is optimal for $CCP_n$.

On the other hand, by Proposition 3.6, the constrained control problem $CCP_n$ can be written in the following alternative form.

CCP'_n: Minimize $\langle \mu, c^0_n \rangle$

subject to: $\mu$ is an occupation measure for $CCP_n$, and

\[ \langle \mu, c^i_n \rangle \leq k^i_n \quad \forall \; i = 1, \ldots, q. \] (3.12)

A measure $\mu \in M(IK)$ is said to be a feasible solution for $CCP'_n$ if it satisfies (3.11)-(3.12), and we denote by $F_n$ the set of feasible solutions for $CCP'_n$. The set of optimal solutions for $CCP'_n$ is denoted by $M_n$. Note that in both cases, bounded and unbounded costs, any $\mu \in F_n$ satisfies that $\langle \mu, c^0_n \rangle < \infty$ (see (4.22)), and hence $\inf CCP'_n < \infty$ for all $n \in \overline{N}$.

**Convergence of values for CCPs.** To obtain our convergence results we impose the following hypothesis.

**Assumption 3.8** (Slater condition) There exist $\pi \in \Pi$ and $\eta > 0$ such that

\[ V^i_\infty(\pi) \leq k^i_\infty - \eta \quad \forall \; i = 1, 2, \ldots, q. \] (3.13)

The condition (3.13) can be obtained via the approximating problems $CCP_n$, see Proposition 3.11.

**Theorem 3.9** (a) Suppose that the Assumptions 3.1(a)-(b) are satisfied, and in addition either Assumption 3.2(a) or 3.3(a)-(b) holds. Then $CCP_n$ — equivalently, $CCP'_n$ — is solvable for each $n \in \overline{N}$. 
(b) Suppose that either Assumptions 3.1, 3.2 and 3.8 or 3.1, 3.3 and 3.8 hold. Then the optimal value of $\text{CCP}_n'$ converges to the optimal value of $\text{CCP}_\infty'$, i.e.

$$\min \text{CCP}_n' \to \min \text{CCP}_\infty' \ \text{as} \ n \to \infty. \quad (3.11)$$

Furthermore, if $\{\mu_n\}$ is a sequence such that $\mu_n \in \mathcal{M}_n$ is an optimal solution for $\text{CCP}_n'$ for each $n \in \mathbb{N}$, then $\{\mu_n\}$ is relatively compact (see Remark 3.4(c)), and any weak accumulation point of $\{\mu_n\}$ is an optimal solution for $\text{CCP}_\infty'$.

(c) Under the Assumptions in part (b), if $\text{CCP}_\infty'$ has a unique optimal solution, say $\mu$, then for any choice of $\mu_n$ in $\mathcal{M}_n$, the sequence $\{\mu_n\}$ converges weakly to $\mu$.

Theorem 3.9 is proved in section 5, as well as Corollary 3.10 and Proposition 3.11, below. It should be noted that part (a) in Theorem 3.9 on the solvability of $\text{CCP}_n$ has been proved by several authors using other approaches; see e.g. [1, 8, 10, 21, 24].

Part (b) of Theorem 3.9 gives the convergence of values of the $\text{CCP}_n$, but we can also get convergence of subsequences of optimal policies for $\text{CCP}_n$.

**Corollary 3.10** Suppose that either Assumptions 3.1, 3.2 and 3.8 or 3.1, 3.3 and 3.8 hold. For each $n \in \mathbb{N}$, let $\varphi_n \in \Phi$ be an optimal policy for $\text{CCP}_n$. Then the sequence $\{\varphi_n\}$ is relatively compact, and any weak accumulation point of $\{\varphi_n\}$ is an optimal policy for $\text{CCP}_n$.

The following proposition gives the (Slater) condition (3.13) using the approximating problems $\text{CCP}_n$.

**Proposition 3.11** Suppose that the Assumptions 3.1(b)-(e) are satisfied, and in addition either Assumption 3.2 or 3.3 holds. Then the following statements (a), (b) and (c) are equivalent.

(a) There exists an integer $N$ for which the following holds: for each $n \geq N$, there is a positive number $\eta_n$ and a policy $\pi \in \Pi$ such that

$$V^i_n(\pi_n) \leq k^i_n - \eta_n \ \forall \ i = 1, 2, \ldots, q, \ (3.15)$$

and $\eta := \inf_{n \geq N} \eta_n > 0$.

(b) There exist positive numbers $N$ and $\eta$ for which the following holds: For each $n \geq N$ there exists $\pi \in \Pi$ such that

$$V^i_n(\pi_n) \leq k^i_n - \eta \ \forall \ i = 1, 2, \ldots, q. \quad (3.16)$$

and $\eta := \inf_{n \geq N} \eta_n > 0$.  

(c) Assumption 3.8 holds.

**Remark 3.12** As a consequence of Proposition 3.11, if the limiting problem $CCP_\infty$ satisfies the Slater condition (Assumption 3.8), then, under the assumptions in Proposition 3.11, (3.16) holds — and hence $CCP_n$ is consistent — for all $n \geq N$. Therefore, the Assumption 3.1(a) in Theorem 3.9(b),(c) and Corollary 3.10 is in fact a redundancy.

**Example 3.13** Consider a discrete-time, single-server queueing system (QS) with service control, in which the state variable denotes the total number of customers in the system at each time $t = 0, 1, \ldots$. At the beginning of a slot, the probability of $x$ customers arriving to the system is $P_x$ for each $x \geq 0$, and if there are customers present, the decision-maker has to choose a service rate from the action set $A := \{a_0, a_1, \ldots, a_N\}$, with $0 = a_0 < a_1 < \ldots < a_N$. If the decision-maker chooses the service rate $a$, the customer will finish service in that slot with probability $a \in A$. Let $\gamma$ be the initial distribution.

The state space is $X = \{0, 1, \ldots\}$, and the transition law is given for $x \geq 1$ and $y \geq -1$ by

$$Q(x + y | x, a) = aP_{y+1} + (1 - a)P_y$$

where $P_{-1} = 0$. There are two costs: the service cost $c^0(x, a) := c(a)$ for all $(x, a) \in IK$, and the holding cost $c^1(x, a) := h(x)$ for all $(x, a) \in IK$, where $h$ is a given bounded function. It is clear that $c^0$, $c^1$, and $Q(y | \cdot)$ for any $y \in X$, are continuous on $IK$.

The decision-maker wishes to minimize the expected discounted service cost, while the expected discounted holding cost is maintained bounded above by a given number $k$.

We may represent this controlled QS as $QS = (c, h, \{P_x\}, \gamma, k)$, and we wish to approximate it with $QS_n = (c_n, h_n, \{P^n_x\}, \gamma_n, k_n)$ satisfying the following conditions.

**Assumption 3.14** $\gamma_n = \gamma$ and $k_n = k$ for all $n \in \mathbb{N}$. Moreover:

(i) $c_n \rightarrow c =: c_\infty$ pointwise on $A$.
(ii) The sequence $\{h_n\}$ is bounded and $h_n \rightarrow h =: h$ pointwise on $X$.
(iii) $P^n_j \rightarrow P_j =: P_j^\infty$ for all $j \in \mathbb{N}$.
(iv) (Slater condition) If $f$ is the deterministic policy that always choose the service rate $a_N$, then $V_\infty^1(f) < k$.

It is clear that Assumption 3.14 implies that Assumptions 3.1(b)-(e), 3.2(a), 3.8 and the alternative condition on the cost $c^1_n$ given in Remark 6.1(b)(ii) and
(b)(iii) hold. On the other hand, the Slater condition in Assumption 3.14(iv) implies Assumption 3.1(a) for each \( n \geq N \) (see Remark 3.12).

Hence, by Theorem 3.9 and Corollary 3.10, we get the solvability of \( QS_n \) for all \( n \geq N \), and the convergence of values and optimal policies.

4 Technical preliminaries

In this section we present some facts that will be used to prove the results stated in section 3. We begin by recalling the following well-known fact on disintegration of p.m.'s (see p. 88 in [7] or p. 89 in [15], for instance).

**Lemma 4.1** If \( \mu \in M^+(K) \) is a p.m., then there exist \( \varphi \in \Phi \) such that

\[
\mu(\{x\} \times C) = \varphi(C|x)\hat{\mu}(x) \quad \forall \ x \in X, \ C \in B(A). \tag{4.1}
\]

Conversely, if \( \varphi \) is in \( \Phi \) and \( \nu \) is a p.m. on \( X \), then the measure

\[
\mu(\{x\} \times C) := \varphi(C|x)\nu(x) \quad \forall \ x \in X \text{ and } C \in B(A) \tag{4.2}
\]

is a p.m. in \( M^+(K) \).

The p.m.'s in (4.1) and (4.2) will be written as \( \mu = \hat{\mu} \circ \varphi \) and \( \mu = \nu \circ \varphi \), respectively. Moreover, for each \( \varphi \in \Phi \) and \( n \in \mathbb{N} \), let

\[
Q_n(\cdot|x, \varphi) := \int_A Q_n(\cdot|a, \varphi) \nu(da|x). \tag{4.3}
\]

This is a (Markov) transition probability and we shall denote by \( Q_n^i(\cdot|x, \varphi) \) the corresponding \( t \)-step transition probabilities.

**Proposition 4.2** Let \( n \in \mathbb{N} \).

(a) For each stationary policy \( \varphi \in \Phi \) there exists a unique occupation measure \( \mu \) for \( CCP_n \) that satisfies

\[
\mu = \hat{\mu} \circ \varphi, \tag{4.4}
\]

\[
V_n^i(\varphi) = \langle \mu, c^i_n \rangle \quad \forall \ i = 0, 1, \ldots, q. \tag{4.5}
\]

(b) For each occupation measure \( \mu \) for \( CCP_n \) there exists a stationary policy \( \varphi \in \Phi \) that is unique \( \hat{\mu} \)-a.e. and satisfies (4.4)-(4.5).
Proof. (a) Choose an arbitrary policy \( \varphi \in \Phi \), and let \( \nu \) be the p.m. on \( X \) defined (using (4.3)) by

\[
\nu(\cdot) := (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t \sum_{y \in X} Q_n^t(\cdot|y, \varphi) \gamma_n(y).
\]  

(4.6)

Then \( \mu := \nu \circ \varphi \) is such that \( \widehat{\mu} = \nu \), and (4.4) follows. On the other hand, (4.6) yields that for each \( \Gamma \) in \( \mathcal{B}(X) \) (and denoting by \( I_{\Gamma} \) the indicator function of \( \Gamma \))

\[
\mu(\Gamma) = \sum_{x \in X} \int_A I_{\Gamma}(x, a) \varphi(da|x) \nu(x)
\]

\[
= (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t \sum_{y \in X} \sum_{x \in X} \int_A I_{\Gamma}(x, a) \varphi(da|x) Q_n^t(x|y, \varphi) \gamma_n(y)
\]

\[
= (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t P_{\gamma_n}^t((x_t, a_t) \in \Gamma).
\]  

(4.7)

This implies (4.5) and also that \( \mu \) is an occupation measure for \( \text{CCP}_n \), i.e., it satisfies (3.10) (see p. 133 of [12]). Finally, to prove the uniqueness of \( \mu \), suppose that \( \mu^* \) is another occupation measure for \( \text{CCP}_n \) that satisfies (4.4)-(4.5). Hence, using (4.4) and (4.3) the expression (3.10) becomes

\[
\widehat{\mu}^*(x) = (1 - \alpha) \gamma_n(x) + \alpha \sum_{y \in X} Q_n(x|y, \varphi) \widehat{\mu}^*(y) \ \forall \ x \in X.
\]

Iterating \( N \) times the latter equality and then letting \( N \to \infty \), we obtain that \( \widehat{\mu}^* \) is precisely the p.m. \( \nu = \widehat{\mu}^* \) in (4.6). Thus \( \mu^* = \mu \).

(b) This follows from Lemma 4.1 and part (a).

Remark 4.3 From the proof of Proposition 4.2 it follows that, for each \( n \in \mathbb{N} \), every occupation measure \( \mu \) for \( \text{CCP}_n \) and its marginal \( \widehat{\mu} \) are of the form (4.7) and (4.6), respectively. In fact, some authors use (4.7) to define the occupation measures (see [12] or [14], for instance).

From the equivalence of (3.8) and (3.9) we immediately get the following.

Lemma 4.4 Let \( \{g_n\} \) be a bounded sequence of functions from \( X \) to \( \mathbb{R} \) such that \( g_n(x) \to g(x) \) for all \( x \in X \), and let \( \{\nu_n\} \) be a sequence in \( P(X) \) such that \( \nu_n \to \nu \) weakly. Then

\[
\sum_{x \in X} g_n(x) \nu_n(x) \to \sum_{x \in X} g(x) \nu(x).
\]  

(4.8)
The following lemma extends (4.8) to spaces more general than $X$. If $S$ is a metric space we denote by $\mathcal{B}(S)$ the Banach space of bounded measurable functions on $S$ with the supremum norm $\| \cdot \|$.

**Lemma 4.5** Let $S$ be a metric space, $\{\nu_n\}$ a sequence in $\mathcal{P}(S)$, and $\{u_n\}$ a sequence in $\mathcal{B}(S)$ such that $u_n \to u$ uniformly, and $\nu_n \to \nu$ weakly.

(a) If $u$ is in $\mathcal{B}(S)$ and it is l.s.c, then

$$\liminf_{n \to \infty} \int_S u_n \, d\nu_n \geq \int_S u \, d\nu.$$ 

(b) If $u$ is in $C_b(S)$, then

$$\lim_{n \to \infty} \int_S u_n \, d\nu_n = \int_S u \, d\nu.$$

**Proof.** (a) As $u$ is (bounded and) l.s.c, the inequality $u_n \geq u - \|u_n - u\|$ yields

$$\liminf_{n \to \infty} \int u_n \, d\nu_n \geq \int ud\nu - \liminf_{n \to \infty} \|u_n - u\| \geq \int ud\nu.$$

(b) If $u$ is in $C_b(S)$, then $u$ and $-u$ are both l.s.c. Hence, applying (a) to $u$ and $-u$ we get (b).

**Lemma 4.6** (a) Let $\{\varphi_n\}$ be sequence in $\Phi$ such that $\varphi_n \to \varphi$ weakly, and $\{\nu_n\}$ a sequence in $\mathcal{P}(X)$ such that $\nu_n \to \nu$ weakly. If $\mu_n := \nu_n \circ \varphi_n$ and $\mu := \nu \circ \varphi$, then $\mu_n \to \mu$ weakly.

(b) Suppose that Assumptions 3.2 holds. If $\{\mu_n\}$ and $\mu$ are as in (a), then

$$\liminf_{n \to \infty} \langle \mu_n, c_i \rangle \geq \langle \mu, c_i \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

(c) If in addition $\varphi_n \equiv \varphi$ for each $n \in \mathbb{N}$ and some $\varphi \in \Phi$, then

$$\lim_{n \to \infty} \langle \mu_n, c_i \rangle = \langle \mu, c_i \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

**Proof.** (a) Pick an arbitrary $h \in C_b(K)$, and define $g_n(x) := \int_{A(x)} h(x,a) \varphi_n(da|x)$ for each $x \in X$. As $h(x, \cdot)$ is in $C_b(A(x))$ for each $x \in X$ and $\varphi_n \to \varphi$ weakly, it follows that $g_n \to g$ pointwise, where $g(x) := \int_{A(x)} h(x,a) \varphi(da|x)$. Moreover, $|g_n(x)| \leq \|h\| < \infty$ for all $n$ and $x$. Therefore, by Lemma 4.4

$$\int h \, d\mu_n = \sum_{x \in X} g_n(x) \nu_n(x) \to \sum_{x} g(x) \nu(x) = \int h \, d\mu,$$

(b) Suppose that Assumptions 3.2 holds. If $\{\mu_n\}$ and $\mu$ are as in (a), then

$$\liminf_{n \to \infty} \langle \mu_n, c_i \rangle \geq \langle \mu, c_i \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

(c) If in addition $\varphi_n \equiv \varphi$ for each $n \in \mathbb{N}$ and some $\varphi \in \Phi$, then

$$\lim_{n \to \infty} \langle \mu_n, c_i \rangle = \langle \mu, c_i \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

**Proof.** (a) Pick an arbitrary $h \in C_b(K)$, and define $g_n(x) := \int_{A(x)} h(x,a) \varphi_n(da|x)$ for each $x \in X$. As $h(x, \cdot)$ is in $C_b(A(x))$ for each $x \in X$ and $\varphi_n \to \varphi$ weakly, it follows that $g_n \to g$ pointwise, where $g(x) := \int_{A(x)} h(x,a) \varphi(da|x)$. Moreover, $|g_n(x)| \leq \|h\| < \infty$ for all $n$ and $x$. Therefore, by Lemma 4.4

$$\int h \, d\mu_n = \sum_{x \in X} g_n(x) \nu_n(x) \to \sum_{x} g(x) \nu(x) = \int h \, d\mu,$$
which yields (a).

Part (b) follows from (a) and Lemma 4.5(a), whereas part (c) follows from the bounded convergence theorem and Lemma 4.4.

By Remark 3.4(b) we may consider the set \( \Phi \) as the product \( \times_{x \in X} \mathcal{P}(A(x)) \). On the other hand, as \( A(x) \) is compact, so is the space \( \mathcal{P}(A(x)) \) with the weak topology. Hence, by the Tychonoff theorem,

\[
\Phi = \times_{x \in X} \mathcal{P}(A(x)) \text{ is compact.} \tag{4.9}
\]

**Lemma 4.7** Suppose that Assumption 3.3 holds. Let \( \{\mu_n\} \subset M^S(K) \) be such that \( \mu_n \) is an occupation measure for \( CCP_n \) for each \( n \in N \) (or \( \mu_n \) is an occupation measure for \( CCP_N \) for each \( n \in N \) and some fixed \( N \in N \)). If \( \mu_n \to \mu \) weakly in \( M^S(K) \), then

(a) \( \lim_{n \to \infty} \langle \mu_n, c^i_n \rangle = \langle \mu, c^i_\infty \rangle \quad \forall \ i = 0, 1, \ldots, q, \)

(b) \( \lim_{n \to \infty} \langle \mu_n, c^m_n \rangle = \langle \mu, c^m_\infty \rangle \quad \forall m \in N, \ i = 0, 1, \ldots, q. \)

**Proof.** We only prove (a) because the proof of (b) is quite similar.

For fixed \( i = 0, 1, \ldots, q \), let \( \{(\mu_m, c^i_m)\} \) be an arbitrary subsequence of \( \{(\mu_n, c^i_n)\} \). Let \( \varphi_m \in \Phi \) such that \( \mu_m = \hat{\mu}_m \circ \varphi_m \) (see Lemma 4.1). By (4.9), there exists a subsequence \( \{\varphi_{m_i}\} \) of \( \{\varphi_m\} \) such that \( \varphi_{m_i} \to \varphi \) weakly in \( \Phi \), and by Lemma 4.6(a) it follows that \( \mu = \hat{\mu} \circ \varphi \). Therefore, for arbitrary \( x \in X \), the Assumptions 3.3(a),(c) yield that the sequences \( \{c^i_m(x, \cdot)\} \) and \( \{\varphi_{m_i}(\cdot | x)\} \) satisfy the hypotheses of Lemma 4.5(b) for each \( i = 0, 1, \ldots, q \). Thus

\[
c^i_{m_i}(x, \varphi_{m_i}) \to c^i_\infty(x, \varphi), \quad \forall \ x \in X, \ i = 0, 1, \ldots, q. \tag{4.10}
\]

Now let \( \nu_0 \) as in Assumption 3.3(b). Then by (3.5), Remark 4.3 and (4.6) we get that

\[
\hat{\mu}_{m_i}(\cdot) \leq \nu(\cdot). \tag{4.11}
\]

Thus from (4.10), (4.11) and Assumption 3.3(b), the Generalized Dominated Convergence Theorem in [13, Theorem 2.2] implies that

\[
\langle \mu_{m_1}, c^i_{m_1} \rangle = \sum_{x \in X} c^i_{m_1}(x, \varphi_{m_1})\hat{\nu}_{m_1}(x) \longrightarrow \sum_{x \in X} c^i_\infty(x, \varphi)\hat{\nu}(x) = \langle \mu, c^i_\infty \rangle.
\]

As the subsequence \( \{(\mu_n, c^i_n)\} \) of \( \{(\mu_n, c^i_n)\} \) was arbitrary, we conclude that every subsequence of \( \{(\mu_n, c^i_n)\} \) has a subsequence that converges to \( \langle \mu, c^i_\infty \rangle \), and (a) follows.
Definition 4.8 Let $\varphi \in \Phi$ be a stationary policy, and $n \in \mathbb{N}$. A measure $\mu \in M(K)$ is said n-equivalent to $\varphi$ if

(a) $\mu$ is an occupation measure for $CCP_n$, and

(b) both $\varphi$ and $\mu$ satisfy (4.4)-(4.5).

On the other hand, if $\mu \in M(K)$ is an occupation measure for $CCP_n$, then a stationary policy $\varphi \in \Phi$ is said to be n-equivalent to $\mu$ if (b) holds.

From the latter definition and Proposition 4.2 it is clear that $\varphi \in \Phi$ is feasible (respectively, optimal) for $CCP_n$ if and only if the measure $\mu$ n-equivalent to $\varphi$ is feasible (respectively, optimal) for $CCP'_n$.

On the other hand, for each stationary policy $\varphi \in \Phi$ (respectively, for each occupation measure $\mu \in M(K)$ for $CCP'_n$), the existence of a measure n-equivalent to $\varphi$ (respectively, the existence of a stationary policy n-equivalent to $\mu$) is guaranteed by Proposition 4.2. We also have the following facts in which we use the concepts in Remark 3.4(c).

Theorem 4.9 Let $\{\varphi_n\} \subset \Phi$ be a sequence of stationary policies and let $\{\mu_n\} \subset P(K)$ be such that $\mu_n$ is n-equivalent to $\varphi_n$ for each $n \in \mathbb{N}$.

(a) Suppose that Assumptions 3.1(b)-(d) hold. If

$$\varphi_n \rightarrow \varphi \ \text{weakly},$$

then there exists $\mu \in P(K)$ for which

$$\mu_n \rightarrow \mu \ \text{weakly}$$

and, in addition, $\mu$ is $\infty$-equivalent to $\varphi$.

(b) Under the same assumptions of part (a), the sequence $\{\mu_n\}$ is relatively compact and, in addition, every weak accumulation point of $\{\mu_n\}$ is an occupation measure for $CCP_\infty$.

(c) Suppose that either Assumptions 3.1(a)-(b) and 3.2(a) or 3.1(a)-(b) and 3.3(a),(b) hold. Then, for each $n \in \mathbb{N}$, the set $\mathcal{F}_n \subset M(K)$ of feasible solutions for $CCP'_n$ is sequentially compact.

(d) Suppose that either set of Assumptions 3.1(b)-(e) and 3.2 or 3.1(b)-(e) and 3.3 hold. For each $n \in \mathbb{N}$, let $\mu_n \in \mathcal{F}_n$. Then $\{\mu_n\}$ is relatively compact and every weak accumulation point of $\{\mu_n\}$ is in $\mathcal{F}_\infty$. 
Proof. (a) Let \( \varphi \) be as in (4.12), and let \( \mu \) be the unique occupation measure \( \text{oo-equivalent to } \varphi \) (see Proposition 4.2(a)). Then part (a) will follow from (4.4), (4.12) and Lemma 4.6(a) if we show that

\[
\hat{\mu}_n \rightharpoonup \text{weakly.} \tag{4.13}
\]

To prove (4.13) we will first show that for each \( y \) and \( x \) in \( X \)

\[
\hat{Q}_n^t(y|x) \rightharpoonup \hat{Q}_\infty^t(y|x) \quad \forall \ t \in \mathbb{N},
\]

(4.14)

where, for notational ease, we write

\[
\hat{Q}_n^t(y|x) := Q_n^t(y|x, \varphi_n) \quad \text{and} \quad \hat{Q}_\infty^t(y|x) := Q_\infty^t(y|x, \varphi).
\]

Take \( t = 1 \), and fix arbitrary \( y \) and \( x \) in \( X \). By the Assumptions 3.1(b),(c) and the weak convergence in (4.12), the sequence \( \{Q_n(y|x, \cdot)\} \) of functions in \( C_b(A(x)) \) and the sequence \( \{\varphi_n(\cdot|x)\} \) of p.m.'s in \( \mathcal{P}(A(x)) \) satisfy the hypotheses of Lemma 4.5(b). Thus

\[
\lim_{n \to \infty} \int_A Q_n(y|x, a) \varphi_n(da|x) = \int_A Q_\infty(y|x, a) \varphi(da|x).
\]

That is

\[
\hat{Q}_n(y|x) \to \hat{Q}_\infty(y|x), \tag{4.15}
\]

which gives (4.14) for \( t = 1 \). Now, to proceed by induction, let us suppose that for each \( y \) and \( x \) in \( X \),

\[
\hat{Q}_n^t(y|x) \rightharpoonup \hat{Q}_\infty^t(y|x) \quad \text{for some } \ t \in \mathbb{N}. \tag{4.16}
\]

Hence, by (4.15), (4.16) and Lemma 4.4, for any fixed \( y \) and \( x \) we get

\[
\lim_{n \to \infty} \sum_{z \in X} \hat{Q}_n^t(y|z) \hat{Q}_n(z|x) = \sum_{z \in X} \hat{Q}_\infty^t(y|z) \hat{Q}_\infty(z|x).
\]

Therefore

\[
\hat{Q}_n^{t+1}(y|x) \to \hat{Q}_\infty^{t+1}(y|x),
\]

which completes the proof of (4.14).

Now, from (4.14) and the Assumption 3.1(d), for each \( t \in \mathbb{N} \) and \( y \in X \), the sequences \( \{\hat{Q}_n^t(y|\cdot)\} \) and \( \{\gamma_n\} \) satisfy the hypotheses of Lemma 4.4. Thus, for each \( t \in \mathbb{N} \),

\[
\lim_{n \to \infty} \sum_{z \in X} \hat{Q}_n^t(y|z) \gamma_n(z) = \sum_{z \in X} \hat{Q}_\infty^t(y|z) \gamma_\infty(z) \quad \forall \ y \in X. \tag{4.17}
\]
Finally, to prove (4.13), choose an arbitrary $y \in X$ and $\epsilon > 0$. Let $l$ be such that $2\alpha^l < \epsilon/2$ for all $t \geq l$. By (4.17), there exists $N_t$ ($t = 0, 1, \ldots, l-1$) such that

$$| \sum_{x \in X} \hat{Q}^l_n(y|x)\gamma_n(x) - \sum_{x \in X} \hat{Q}^l_\infty(y|x)\gamma_\infty(x) | < \epsilon/2,$$

for all $n \geq N_t$. By (4.6) and the Remark 4.3, taking $N := \max\{N_0, \ldots, N_{l-1}\}$ we get that for all $n \geq N$

$$|\hat{\mu}_n(y) - \hat{\mu}(y)| \leq (1 - \alpha) \sum_{t=0}^{l-1} \alpha^t \sum_{x \in X} \hat{Q}^l_n(y|x)\gamma_n(x) - \sum_{x \in X} \hat{Q}^l_\infty(y|x)\gamma_\infty(x) | \leq (1 - \alpha^l)\epsilon/2 + 2\alpha^l \leq \epsilon.$$

This implies that $\hat{\mu}_n(y) \to (y)$ for all $y \in X$, which gives (4.13); see (3.8).

(b) Let $\varphi_n \in \Phi$ and $\mu_n \in \mathbb{P}(\mathbb{K})$ be such that $\mu_n$ is $n$-equivalent to $\varphi_n$ for each $n \in \mathbb{N}$. To prove (h), let $\{\mu_m\}$ an arbitrary subsequence of $\{\mu_n\}$. By (4.9), there exists a subsequence $\{\varphi_{m_l}\}$ of $\{\varphi_m\}$ that satisfies (4.12), i.e.

$$\varphi_{m_l} \to \varphi \text{ weakly} \quad (4.18)$$

for some $\varphi \in \Phi$. Now, from (4.18) and the uniqueness of the measure $\mu_{m_l}$ $m_l$-equivalent to $\varphi_{m_l}$ for each $m_l \in \mathbb{N}$ (see Proposition 4.2(a)), from part (a) we get

$$\mu_{m_l} \to \mu \text{ weakly} \quad (4.19)$$

where $\mu$ is $\infty$-equivalent to $\varphi$, that is, $\mu$ is an occupation measure for CCP. Finally, by (4.19), $\{\mu_n\}$ is relatively compact.

(c) Let $n \in \mathbb{N}$ be fixed and let $\{\mu_l\} \subset \mathcal{F}_n$. For each $l \in \mathbb{N}$, let $\varphi_l \in \Delta_n$ be a stationary policy, $n$-equivalent to $\mu_l$. As $\Phi$ is compact (see (4.9)), the sequence $\{\varphi_l\} \subset \Phi$ has a subsequence $\{\varphi_m\}$ such that $\varphi_m \to \varphi$ weakly for some $\varphi \in \Phi$. By the uniqueness of the measures $n$-equivalent to $\varphi_m$, proceeding as in the proof of (a) we get that $\mu_m \to \mu$ weakly, where $\mu$ is $n$-equivalent to $\varphi$, that is, $\mu$ satisfies (a) and (b) in Definition 4.8. Hence, it only remains to show that $\mu$ satisfies the constraints (3.12). For the bounded case, (4.19) and Assumption 3.2(a), together with Lemma 4.6(b), yield

$$\langle \mu, c^i_n \rangle \leq \liminf_{m \to \infty} \langle \mu_m, c^i_n \rangle \leq k^i_n \quad \forall \; i = 1, 2, \ldots, q. \quad (4.20)$$
Therefore \( \mu \) is in \( \mathcal{F}_n \). Similarly, for the unbounded case we get (4.20) from the Assumption 3.3 and Lemma 4.7(b).

(d) For each \( n \in \mathbb{N} \), let \( \mu_n \in \mathcal{F}_n \). By part (b), \( \{\mu_n\} \) is relatively compact. Now let \( \mu \) be a weak accumulation point of \( \{\mu_n\} \). By (b) again, \( \mu \) satisfies (3.11) with \( n = \infty \). Thus, to end the proof, we only need to show that \( \mu \) satisfies the constraints (3.12), which is obtained as (4.20). Namely, for the bounded case, Assumption 3.2 together with Lemma 4.6(b) imply that

\[
\langle \mu, c^i_n \rangle \leq \liminf_{n \to \infty} \langle \mu_n, c^i_n \rangle \leq \lim_{n \to \infty} k^i_n = k^i_\infty \quad V i = 1, 2, \ldots, q.
\]

(4.21)

Therefore \( \mu \) is in \( \mathcal{F}_\infty \). In the unbounded case, (4.21) follows from Assumption 3.3 and Lemma 4.7(a).

Remark 4.10 At the end of the proofs for parts (c) and (d) of Theorem 4.9, it can be seen that the bounded and unbounded cost cases can be dealt with similarly, the only one difference in the proofs being that instead of Lemma 4.6(b), for the unbounded case is used Lemma 4.7. In fact, the unbounded case behaves as the bounded case, in the sense that for each \( i = 0, 1, \ldots, q \) and \( n \in \mathbb{N} \)

\[
\langle \mu, c^i_n \rangle < \infty \quad \text{for each occupation measure } \mu \text{ for } CCP_n.
\]

(4.22)

Indeed, let \( \mu \) be an occupation measure for \( CCP_n \) and \( \varphi \in \Phi \) a stationary policy \( n \)-equivalent to \( \mu \). From (3.5), Remark 4.3 and (4.6) we get that \( \hat{\mu}(\cdot) \leq \nu_0(\cdot) \) and thus \( \mu(\cdot) \leq \nu_0 \circ \varphi(\cdot) \). Using the latter fact and the conditions (3.4) and (3.6) we get

\[
\langle \mu, c^i_n \rangle \leq \langle \nu_0 \circ \varphi, g \rangle = \sum_{x \in \mathcal{X}} g(x)\nu_0(x) < \infty.
\]

5 Proof of the main results

In the main results of section 3, we have considered two cases, the bounded and unbounded cost cases. However the proofs in the present section will be done only for the bounded case, because the proofs for the unbounded case are essentially the same; see Remark 4.10. More precisely, in the unbounded case we have to use Lemma 4.7 instead of Lemmas 4.4 and 4.6(b)-(c) in all parts where these lemmas are used (see (5.3), (5.4), (5.8), (5.10), (5.12), and (5.13)).

To prove Theorem 3.9 we first state two lemmas that show the connection between that theorem and Theorem 2.4.
Lemma 5.1 Suppose that the Assumptions 3.1(b)-(e) and 3.8 are satisfied, and in addition either Assumption 3.2 or 3.3 holds. Let $\mu \in \mathcal{F}_\infty$. Then there exists an integer $N$ and feasible solutions $\mu_n \in \mathcal{F}_n$ for $n \geq N$ such that $\mu_n \to \mu$ weakly and

$$\lim_{n \to \infty} \langle \mu_n, c^0_n \rangle = \langle \mu, c^0_\infty \rangle. \quad (5.1)$$

Proof. Let $\mu$ be in $\mathcal{F}_\infty$. We shall construct the desired sequence $\{\mu_n\}$ in three steps. First, we give a sequence $\{\delta_n\} \subset \mathcal{M}(K)$ such that for each $n \in \mathbb{N}$, $\delta_n$ is an occupation measure for $CCP_n$ and

$$\delta_n \to \mu \quad \text{weakly.} \quad (5.2)$$

However, we cannot guarantee that $\delta_n$ is in $\mathcal{F}_n$ for all $n \in \mathbb{N}$, and, therefore, in the second step we introduce a "correcting" sequence $\{\theta_n\}$ such that $\{\theta_n\}$ is convergent and, for each $n \in \mathbb{N}$, $\delta_n$ is in $\mathcal{F}_n$. Finally, in the third step we construct the desired sequence $\{\mu_n\}$ as a particular convex combination of the sequences $\{\delta_n\}$ and $\{\theta_n\}$.

Step 1. Let $\varphi \in \Phi$ be a stationary policy $\infty$-equivalent to $\mu$. Now let $\{\delta_n\}$ be a sequence of measures in $\mathcal{M}(K)$ such that $\delta_1$ is $n$-equivalent to $\varphi$ for each $n \in \mathbb{N}$ (see Proposition 4.2(a)). Thus, as $\varphi_n \equiv \varphi$ satisfies (4.12), from Theorem 4.9(a) we get (5.2). In turn (5.2) yields the convergence of the marginals, i.e., $\hat{\delta}_n \to \hat{\mu}$ weakly. Moreover, using the Assumption 3.2 and Lemma 4.6(c) we get that, as in (4.21),

$$\lim_{n \to \infty} \langle \delta_n, c^i_n \rangle = \langle \mu, c^i_\infty \rangle \leq k^i_\infty \quad \forall i = 1, 2, \ldots, q. \quad (5.3)$$

Step 2. By Corollary 3.7, the (Slater) condition (3.13) also holds for some stationary policy $\varphi' \in \Phi$. For each $n \in \mathbb{N}$, let $\theta_n$ be the measure $n$-equivalent to $\varphi'$. Then Theorem 4.9(a) yields that $\theta_n \to \theta$ weakly, where $\theta$ is the measure $\infty$-equivalent to $\varphi'$. Hence, as in Step 1, Lemma 4.6(c) implies that

$$\lim_{n \to \infty} \langle \theta_n, c^i_n \rangle = \langle \theta, c^i_\infty \rangle = V^i_\infty(\varphi') \leq k^i_\infty - \eta \quad \forall i = 1, 2, \ldots, q. \quad (5.4)$$

Let $\epsilon$ be such that $0 < 4\epsilon < \eta$. From (5.3), (5.4) and the Assumption 3.1(d), there exist $N_\epsilon$ such that for all $n \geq N_\epsilon$,

$$\langle \theta_n, c^i_n \rangle \leq k^i_\infty - \eta + \epsilon,$n \geq N_\epsilon$,

$$\langle \delta_n, c^i_n \rangle \leq k^i_\infty + \epsilon,$$

$$|k^i_n - k^i_\infty| \leq \epsilon. \quad (5.5)$$
Now let $\delta_{\lambda}^n := \lambda \theta_n + (1 - \lambda) \delta_n$ for all $n \geq N$, $\lambda \in [0, 1]$, and
\[
\lambda_\varepsilon := \frac{\varepsilon + (k_{n} - k_{n}^i)}{\eta} \quad (5.6)
\]
Then $\lambda_\varepsilon < (2\varepsilon/\eta) < 1/2$, and for each $n \geq N$, and $\lambda \in [\lambda_\varepsilon, 1]$,
\[
\begin{align*}
\langle \delta_{\lambda_\varepsilon}^n, c_n^i \rangle &= \lambda \langle \theta_n, c_n^i \rangle + (1 - \lambda) \langle \delta_n, c_n^i \rangle \\
&\leq \lambda (k_{n}^\infty - \eta + \varepsilon) + (1 - \lambda) (k_{n}^\infty + \varepsilon) \\
&\leq k_{n}^i.
\end{align*}
\]
Hence, $\delta_{\lambda_\varepsilon}^n$ is in $\mathcal{F}_n$ for all $n \geq N$, and $\lambda \in [\lambda_\varepsilon, 1]$. Furthermore, by (5.5) and (5.6), $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

**Step 3.** We are finally ready for the construction of the desired sequence $\mu_n$. Let $\{\varepsilon_l\} \subset \mathbb{R}$ be a decreasing sequence such that $\varepsilon_l \downarrow 0$ and $0 < \varepsilon_l < \eta/4$. As in the previous paragraph, for each $\varepsilon_l$ there exist numbers $N_l$ (which we can suppose that form an increasing sequence $\to \infty$) and $\lambda_{N_l}$ such that
(i) $\delta_{\lambda_{N_l}}^n$ is in $\mathcal{F}_n \forall n \geq N_l$, and
(ii) $\lambda_{N_l} \to 0$ when $l \to \infty$.

We take $\mu_n := \delta_{\lambda_{N_l}}^n$ for all $N_l \leq n < N_{l+1}$. Thus the sequence $\{\mu_n\}$ satisfies that, by (i), $\mu_n$ is in $\mathcal{F}_n$ for all $n \geq N_1$ and, by (ii),
\[
\mu_n \to \mu \quad \text{weakly}.
\quad (5.7)
\]
Finally, to get (5.1) note that Lemma 4.6(c) yields
\[
\lim_{n \to \infty} \langle \theta_n, c_n^0 \rangle = (0, c_0^0) \quad \text{and} \quad \lim_{n \to \infty} \langle \delta_n, c_n^0 \rangle = (\mu, c_0^0).
\quad (5.8)
\]
From (5.7) and (5.8) we obtain (5.1). \hfill \blacksquare

**Lemma 5.2** Either set of Assumptions 3.1(b)-(e), 3.2 and 3.8 or 3.1(b)-(e), 3.3 and 3.8 imply the Assumption 2.1.

**Proof.** Theorem 4.9(d) implies (2.1), whereas Lemma 4.6(b) gives (2.2). Finally, part (c) of Assumption 2.1 follows from Lemma 5.1. \hfill \blacksquare

**Proof of Theorem 3.9.** (a) Fix an arbitrary $n \in \mathbb{N}$. To prove the solvability of $\text{CCP}_n$ it suffices to show the solvability of $\text{CCP}_n'$. Let $\{\mu_m\} \in$
\( M^+(K) \) be a minimizing sequence for \( \text{CCP}'_n \), that is, \( \mu_m \) is in \( \mathcal{F}_n \) for each \( m \in \mathbb{N} \) and, in addition,

\[
\langle \mu_m, c_n^0 \rangle \downarrow \inf \text{CCP}'_n \quad \text{as} \quad m \to \infty,
\]

where \( \inf \text{CCP}'_n \) stands for the value of \( \text{CCP}'_n \). As \( \mathcal{F}_n \) is sequentially compact (see Theorem 4.9(c)), there exists a subsequence \( \{\mu_l\} \) of \( \{\mu_m\} \) and \( \mu \) in \( \mathcal{F}_n \) such that \( \mu_l \to \mu \) weakly. As \( c_n^0 \) is l.s.c and bounded below (see Assumption 3.2(a)), (5.9) implies

\[
\langle \mu, c_0^0 \rangle \leq \liminf_{l \to \infty} \langle \mu_l, c_n^0 \rangle = \inf \text{CCP}'_n.
\]

Thus \( \mu \) is an optimal solution for \( \text{CCP}'_n \).

(b) For each \( n \in \mathbb{N} \), let \( \mu_n \in \mathcal{P}(K) \) be an optimal solution for \( \text{CCP}'_n \).

From Theorem 4.9(d), \( \{\mu_n\} \) is relatively compact, and from Theorem 2.3(a), any weak accumulation point of \( \{\mu_n\} \) is an optimal solution for \( \text{CCP}'_n \). To prove the convergence (3.14), let \( \{\min \text{CCP}'_m\} \) be a subsequence of the bounded sequence of real numbers \( \{\min \text{CCP}'_n\} \). From Theorems 4.9(d) and 2.3(a) there exists a subsequence \( \mu_m_i \) of \( \mu_m \) that converges weakly to some \( \mu \in \mathcal{P}(K) \) and

\[
\min \text{CCP}'_m = \langle \mu_m_i, c_0^0 \rangle \quad \langle \mu, c_0^0 \rangle = \min \text{CCP}'_n.
\]

Since the subsequence \( \{\min \text{CCP}'_m\} \) of \( \{\min \text{CCP}'_n\} \) was arbitrary, (3.14) follows.

(c) For each \( n \in \mathbb{N} \), choose an arbitrary optimal solution \( \mu_n \) for \( \text{CCP}'_n \). (The existence of such a \( \mu_n \) is ensured by part (a).) By (b), \( \{\mu_n\} \) is relatively compact, and so it has a subsequence \( \{\mu_m\} \) that converges weakly to an optimal solution for \( \text{CCP}'_n \). Moreover, \( \mu' = \mu \) (by the uniqueness of \( \mu \)). A similar argument shows that \( \mu_m \to \mu \) weakly for any convergent subsequence \( \{\mu_m\} \) of \( \{\mu_n\} \) and, therefore, \( \mu_n \to \mu \) weakly (see [4, Theorem 2.31]).

Proof of Corollary 3.10. For each \( n \in \mathbb{N} \), let \( \varphi \in \Phi \) be an optimal policy for \( \text{CCP}_n \). By the compactness of \( \Phi \) it follows that \( \{\varphi_n\} \) is relatively compact. Let \( \{\varphi_m\} \) be subsequence of \( \{\varphi_n\} \) such that \( \varphi_m \to \varphi \) weakly, and let \( \mu_m \) be \( m \)-equivalent to \( \varphi_m \). By Theorem 4.9(a), there exists \( \mu \in M(\mathbb{I}K) \) such that \( \mu_m \to \mu \) and \( \mu \) is \( \infty \)-equivalent to \( \varphi \). Since \( \varphi_m \) is an optimal strategy for \( \text{CCP}_n \), also \( \mu_m \) is an optimal solution for \( \text{CCP}'_m \). Hence, by Theorem 3.9(b), \( \mu \) is an optimal solution for \( \text{CCP}'_n \) and, since \( \varphi \) is \( \infty \)-equivalent to \( \mu \), it follows that \( \varphi \) is an optimal policy for \( \text{CCP}_n \).

Proof of Proposition 3.11. The equivalence \( (a) \Leftrightarrow (b) \) is immediate.

(\( b \Rightarrow (c) \). By Corollary 3.7, for each \( n \in \mathbb{N} \), there exists a stationary policy \( \varphi_n \) that satisfy (3.16). For each \( n \in \mathbb{N} \), let \( \mu_n \) be the measure \( n \)-equivalent to
As \( \varphi_n \) is feasible for CCP, the p.m. \( \mu_n \) is in \( \mathcal{F}_n \). By Theorem 4.9(d), there exist a subsequence \( \{ \mu_m \} \) of \( \{ \mu_n \} \) and \( \mu \in \mathcal{F}_\infty \), such that
\[
\mu_m \to \mu \quad \text{weakly.} \tag{5.11}
\]
Let \( \varphi \) be \( \infty \)-equivalent to \( \mu \). By (5.11), and using Assumption 3.2, Lemma 4.6(b) yields
\[
V^i_\infty(\varphi) = \langle \mu, c^i_\infty \rangle \leq \liminf_{m \to \infty} \langle \mu_m, c^i_m \rangle. \tag{5.12}
\]
Finally, using (3.12) and Assumption 3.1(e),
\[
V^i_\infty(\varphi) \leq \lim_{m \to \infty} (k^i_m - \eta) = k^i_\infty - \eta \quad \forall \ i = 1, 2, \ldots, q.
\]
Hence \( \pi = \varphi \) satisfies the (Slater) condition (3.13).

(c) \( \Rightarrow \) (b). By Corollary 3.7, the (Slater) condition (3.13) also holds for some stationary policy \( \varphi \in \Phi \). For each \( n \in \mathbb{N} \), let \( \mu_n \in M(K) \) be \( n \)-equivalent to \( \varphi \). By Theorem 4.9(a), \( \mu_n \to \mu \) weakly where \( \mu \) is \( \infty \)-equivalent to \( \varphi \). Then Lemma 4.6(e) and (3.13) imply
\[
\lim_{n \to \infty} \langle \mu_n, c^i_n \rangle = \langle \mu, c^i_\infty \rangle \leq k^i_\infty - \eta. \tag{5.13}
\]
Let \( 0 < \epsilon < \eta/4 \). By (5.13) and Assumption 3.1(e), there exists \( N \in \mathbb{N} \) such that, for all \( n \geq N \), \( k^i_\infty \leq k^i_\infty + \epsilon \) and \( \langle \mu_n, c^i_n \rangle \leq k^i_\infty - \eta + \epsilon \) for all \( i = 1, 2, \ldots, q \). Therefore, for each \( i = 1, 2, \ldots, q \),
\[
\langle \mu_n, c^i_n \rangle \leq k^i_n - \eta/2 \quad \forall \ n \geq N,
\]
which implies that \( \varphi \) satisfies the Slater condition
\[
V^i_n(\varphi) \leq k^i_\infty - \eta/2 \quad \forall \ n \geq N. \quad \blacksquare
\]

6 Further remarks

Theorem 3.9 and Corollary 3.10 give results on the convergence of the optimal values and optimal policies for a class of constrained Markov control processes. These results are essentially a consequence of Theorem 2.3, which concerns general minimization problems. In this final section we show that some of the hypotheses on Theorem 3.9 and Corollary 3.10 can be weakened and/or adapted to other classes of CCPs.
Remark 6.1 From the proof of Theorem 3.9 and Corollary 3.10 it can be seen the following.

(a) Assumptions 3.1(c) and 3.2(b) on the convergence of the transition laws $Q_n$ and the costs $c_n^i$, respectively, can be replaced with the following conditions.

(i) For each sequence $\{\varphi_n\} \subset \Phi$ that converges weakly to some $\varphi \in \Phi$ we have that

$$\lim_{n \to \infty} Q_n(y|x, \varphi_n) = Q_\infty(y|x, \varphi) \quad \forall \ y, x \in X.$$ 

(ii) $c_n^i \to c^i_\infty$ pointwise for each $i = 0, 1, \ldots, q$, and, moreover, for each sequence of feasible solutions $\mu_n \in \mathcal{F}_n$ that converges weakly to some $\mu \in \mathbf{P}(K)$ we have that

$$\liminf_{n \to \infty} \langle \mu_n, c_n^i \rangle \geq \langle \mu, c^i_\infty \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

Of course, Assumptions 3.1(b)-(c) imply the condition (i) (see Lemma 4.5(b)), whereas the Assumptions 3.2 yield the condition (ii) (see Lemma 4.5(a)).

(b) As a consequence of part (a), the Assumption 3.2 for the bounded cost case can be replaced with the following alternative conditions.

(i) Assumption 3.2(a) holds, and

(22) the cost $c^i_\infty(x, \cdot)$ is continuous on $A(x)$ for each $i = 0, 1, \ldots, q$ and $x \in X$.

(iii) $c^i_n(x, a) \to c^i_\infty(x, a)$ for each $(x, a) \in K$ and $i = 0, 1, \ldots, q$, and the convergence is uniform on $A(x)$.

Indeed, the latter conditions imply that for any sequence $\{\mu_n\} \in \mathbf{P}(K)$ such that $\mu_n \to \mu$ weakly, we get

$$\liminf_{n \to \infty} \langle \mu_n, c^i_n \rangle = \langle \mu, c^i_\infty \rangle \quad \forall \ i = 0, 1, \ldots, q.$$ 

(6.1)

The proof of (6.1) can be obtained as the proof of Lemma 4.7 but using Lemma 4.4 instead the Generalized Dominated Convergence Theorem in [13].

Countable action sets. Some of our proofs above depend on the compactness of $\Phi$ (see (4.9)), but similar arguments give results for the countable action set case.

Consider the constrained control model

$$\{X, A, \{A(x) \mid x \in X\}, Q, c^0, \gamma, c, k\},$$

(6.2)

which is as in (3.1) except that $A$ is countable.
Assumption 6.2  (i) $Q_n(y|x,a) \to Q_\infty(y|x,a)$ $\forall$ $y \in X, (x,a) \in I\K$.  
(ii) for each $i = 0,1,\ldots,q$ the sequence of costs $\{c^i_n\}$ is bounded, and $c^i_n(x,a) \to c^i_\infty(x,a)$ $\forall$ $(x,a) \in I\K$.  
(iii) $\gamma_n \to \gamma_\infty$ weakly.  
(iv) $k^i_n \to k^i_\infty$.  
(v) Assumption 3.8 holds.

Under this assumption Theorem 3.9 becomes as follows.

Proposition 6.3  Suppose that $A$ is countable and that Assumption 6.2 holds.  
For each $n \in \mathbb{N}$, let $\varphi_n$ be an optimal policy for $CCP_n$ (if it exists).  Then any weak accumulation point of $\{\varphi_n\}$ is an optimal policy for $CCP_n$.  In other words, if there exists a subsequence $\{\varphi_m\}$ of $\{\varphi_n\}$ and $\varphi \in \Phi$ such that $\varphi_m \to \varphi$ weakly, then $\varphi$ is optimal for $CCP_n$.  Furthermore, the optimal values of $CCP_m$ converge to the optimal value of $CCP_n$, i.e.,  

$$\min CCP_m \to \min CCP_n.$$  

Example 6.4  Approximation of the state and action sets.  Consider the constrained control model (6.2), and suppose that we wish to approximate the state and action sets by finite subsets as follows.

Assumption 6.5  (a) There exist nondecreasing sequences $\{X_n\}, \{A_n\}$ of finite subsets of $X$ and $A$, respectively, such that $\bigcup X_n = X$, $\bigcup A_n = A$.  Moreover, $A_n(x) := A(x) \cap A_n$, $\# \$, $Q(X_n|x,a) > 0$, and $\gamma(X_n) > 0$ for each $n \in \mathbb{N}, x \in X$ and $a \in A_n(x)$.  
(b) Assumption 3.8 holds.

Recall that $I_D$ denotes the indicator function of the set $D$, and for each $n \in \mathbb{N}$ let $I\K_n := \{(x,a)|x \in X_n, a \in A_n(x)\}$,  

$$Q_n(y|x,a) := \frac{Q(y|x,a)}{Q(X_n|x,a)} I\K_n(x,a) \forall (x,a) \in I\K, \quad (6.3)$$  
$$\gamma_n(x) := \frac{\gamma(x)}{\gamma(X_n)} I_{X_n}(x) \forall x \in X, \text{ and} \quad (6.4)$$  
$$c^i_n := c^i I_{I\K_n} \forall i = 0,1,\ldots,q. \quad (6.5)$$  

We thus obtain the sequence of approximating constrained control models  

$$(X,A, \{A(x)|x \in X\}, Q_n,c^0_n,\gamma_n,c_n,k) \quad (6.6)$$  

which can be viewed, of course, as approximations to (6.2) by finite state and action sets.

From Assumption 6.5 and (6.3)-(6.5) it is clear that the sequence in (6.6) satisfies the Assumption 6.2, and hence the conclusions in Proposition 6.3.
References


