Takagi–Sugeno Fuzzy Scheme for Real-Time Trajectory Tracking of an Underactuated Robot

Ofelia Begovich, Edgar N. Sanchez, Senior Member, IEEE, and Marcos Maldonado

Abstract—This contribution presents an approach to achieve trajectory tracking for nonlinear systems. Combining linear regulator theory with the Takagi–Sugeno (T–S) fuzzy methodology; an algorithm is described for this tracking. The main contribution of the paper consists in the real-time application of this algorithm to a specific underactuated robot: the Pendubot.

Index Terms—Fuzzy control, linear regulator theory, nonlinear system, Takagi–Sugeno (T–S) approach, trajectory tracking.

I. INTRODUCTION

RECENTLY, the Takagi–Sugeno (T–S) fuzzy approach [1] has been used to model nonlinear systems. Using this technique, the dynamic of a nonlinear system is easily obtained by linearization near different operation points or (if this model is not well known) by input–output identification around these points. Once these linear models obtained, local linear controller can be designed; the overall controller would be a fuzzy blending of these controllers. Lately several algorithms of fuzzy T–S control have been developed. In [2], an iterative procedure for designing stable T–S fuzzy control is proposed, which is based on finding a common positive definite matrix satisfying the associated Lyapunov equations of local models. To avoid this iterative process, synthesis of fuzzy T–S controllers is directly obtained by minimization of a linear performance index under linear matrix inequalities (LMIs) constrains; solving this optimization problem, the resulting fuzzy controller exhibits stability and performance [3]–[5]. For all these references, examples are limited to set-point regulation.

Nonlinear system trajectory tracking based on T–S fuzzy approach, which is more difficult than set-point regulation, has been less analyzed. In [6], tracking is achieved by minimizing the error between the nonlinear system and a nonlinear reference model, both of them modeled using T–S method; then the error is linearized and closed-loop stability conditions are established by means of LMIs. Simultaneously, we proposed an alternative approach for the trajectory tracking for nonlinear systems [7]. This work combines linear regulator theory [8] with the T–S fuzzy method and can be applied to multiple input–multiple output (MIMO) systems. Viability of this approach was illustrated with real-time results for the level control of a two tanks prototype. Then, we considered the application of this scheme to an underactuated mechanism; a preliminary version of the respective version was presented in [9]. Recently, a comprehensive publication on T–S fuzzy schemes for trajectory tracking was carried out in [10]. It is worth mentioning that the results in [7], [9], were obtained independently from [10]. To achieve trajectory tracking, a T–S fuzzy method is proposed in [11], where, the local linear controllers are designed to minimize an $H_{\infty}$ performance index. Other related publications about trajectory tracking for nonlinear systems using T–S fuzzy approach are [12] for the single input–single output (SISO) case and [13] for the MIMO one; there, control laws, combining feedback linearization, adaptive control and sliding modes, are implemented. In both [12] and [13], the T–S fuzzy systems are used to approximate either the plant or the controller. Evaluation of the proposed schemes in [6] and [10]–[13] are limited to simulation results.

This paper is an enlarged version of [9]. Underactuated mechanical systems posses fewer actuators than degrees of freedom. Complex internal dynamics, nonholonomic behavior and lack of feedback linearizability are often exhibited in such systems, making the class a challenging one for synthesis of control schemes. The underactuated system, we consider is a two-link robot called the Pendubot, which is used for research in nonlinear control, as well as for education in various concepts like nonlinear dynamics and robotics. This device is a two-link planar robot with actuator at the shoulder (link 1) and no actuator at the elbow (link 2), which moves freely around link 1 [14]. Linear and nonlinear algorithms has been frequently used to stabilize this system in its vertical position [14], [15]. The fact that the Pendubot is not right invertible makes it difficult the task of trajectory tracking; hence, as far as we know, there are few publications related to this topic. In [16] a sinusoidal signal of small amplitude is tracked using a nonlinear algorithm based in the regulator theory. Our main contribution is the experimental verification of the scheme we proposed in [7] and [9], which is also included in [10]. As testbed, we use a complex nonlinear system: the Pendubot. Below, we discuss how the proposed controller improves existing experimental results, regarding trajectory tracking for this system.

This paper is organized as follows: the T–S fuzzy scheme is presented in Section II. In Section III, the real-time experimental application is discussed. Finally, conclusions are stated.

II. T–S FUZZY CONTROL ALGORITHM

In this section, we present the used T–S fuzzy control scheme for nonlinear system trajectory tracking, as well as some well-known preliminaries about T–S model and linear regulator theory.
A. **T–S Fuzzy Model**

In this approach, nonlinear systems are approximated by a set of linear local models. A dynamic T–S fuzzy model [1] is described by a set of “IF–THEN” rules as follows:

\[ \text{\textit{ith Rule:}} \]

\[
\begin{align*}
\text{IF} \quad & z_i(t) \text{ is } M_{ik}, \ldots, z_j(t) \text{ is } M_{ji}, \ldots, z_q(t) \text{ is } M_{qi} \\
\text{THEN} \quad & \begin{cases} 
\dot{x}(t) = A_{ij}x(t) + B_{ij}u(t) \\
y(t) = C_{ij}x(t)
\end{cases}
\end{align*}
\]

where

- \(i = 1, \ldots, r\) with \(r\) the number of rules;
- \(z_j(j = 1, \ldots, q)\) premise variables, which may be functions of the states or other variables;
- \(M_{ij}\) fuzzy sets;
- \(x \in \mathbb{R}^n\) state vector;
- \(u \in \mathbb{R}^m\) input vector;
- \(y \in \mathbb{R}^p\) output vector;
- \(A_{ij}, B_{ij}, C_{ij}\) matrices of adequate dimension.

We will note \(z(t)\) as the vector containing all the \(z_j\).

The final state and output of the fuzzy system is inferred as:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(z(t)) \{ A_{ij}x(t) + B_{ij}u(t) \} \\
y(t) &= \sum_{i=1}^{r} \lambda_i(z(t)) C_{ij}x(t)
\end{align*}
\]  \(\text{(1)}\)

where \(\lambda_i(z(t)) = \prod_{j=1}^{q} \mu_{ij}(z_j)\) and \(\mu_{ij}\) is the membership function of \(z_j(t)\) in \(M_{ij}\).

B. **Linear Regulator**

In linear regulator theory [8], the control goal is to obtain a stable closed-loop system and asymptotic tracking error, for every possible initial state and every possible exogenous input in a prescribed family of functions of time. This latter requirement is also know as the property of “output regulation.”

Let consider a linear system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + Pw(t) \\
\dot{u}(t) &= Sw(t) \\
\dot{e}(t) &= Cx(t) + Qw(t)
\end{align*}
\]  \(\text{(2)}\)

where

- \(x\) internal state vector of the plant;
- \(u\) control input vector;
- \(w\) vector containing external disturbances and/or references;
- \(e\) tracking error.

To solve this problem, the following is assumed.

- **A1)** The pair \((A, B)\) is stabilizable.
- **A2)** The exosystem is antistable, i.e., all the eigenvalues of \(S\) have nonnegative real part.

Assuming A1) and A2), the problem of output regulation via state feedback can be determined if and only if there exist matrices \(\Pi\) and \(\Gamma\), which solve the following linear matrix equations:

\[
\begin{align*}
\Pi A + A \Pi + B \Gamma + \Gamma B &= 0 \\
\Pi C &= 0
\end{align*}
\]  \(\text{(3)}\)

The control law is given as

\[
u(t) = Kx(t) + Lw(t),\]

where \(K\) is any matrix such that \((A + BK)\) is Hurwitz. Defining \(L = \Gamma - \Pi K\) we have

\[
u(t) = Kx(t) + Lu(t),\]

C. **T–S Algorithm**

At this stage, we present the used T–S fuzzy control for nonlinear system trajectory tracking. For the controller rules, we use as antecedents the same fuzzy sets used in the plant rules. As consequents, we design local control laws based on the linear regulator theory, for each local linear model. The rules for the plant and controller are

\[ \text{\textit{ith Plant Rule:}} \]

\[
\begin{align*}
\text{IF} \quad & z_i(t) \text{ is } M_{ik}, \ldots, z_j(t) \text{ is } M_{ji}, \ldots, z_q(t) \text{ is } M_{qi} \\
\text{THEN} \quad & \begin{cases} 
\dot{x}(t) = A_{ij}x(t) + B_{ij}u(t) + P_{ij}w(t) \\
\dot{w}(t) = Sw(t) \\
\dot{e}(t) = C_{ij}x(t) + Q_{ij}w(t)
\end{cases}
\end{align*}
\]

\[ \text{\textit{ith Controller Rule:}} \]

\[
\begin{align*}
\text{IF} \quad & z_i(t) \text{ is } M_{ki}, \ldots, z_j(t) \text{ is } M_{kj}, \ldots, z_q(t) \text{ is } M_{qi} \\
\text{THEN} \quad & u = K_{ij}x(t) + L_{ij}w(t)
\end{align*}
\]

where \(K_{ij}\) is any matrix such that \((A_i + B_iK_{ij})\) is Hurwitz; \(i, j, k\) satisfy (3) for each \((A_i, B_i, C_i, P_i, S_i, Q_i), i = 1, \ldots, r\).

The output of the fuzzy controller is given as

\[
u(t) = \sum_{i=1}^{r} \lambda_i(z(t)) [K_{ij}x(t) + L_{ij}w(t)]
\]

Regarding stability analysis for this scheme, we refer the reader to [10].
III. EXPERIMENTAL SETUP

In the following, we describe the experimental setup.

A. Pendubot Description

The Pendubot, in our laboratory, consists of two rigid aluminum links. The first link is coupled to a dc motor mounted to a base. Link 2 is coupled to the end of the link 1 by means of an underactuated joint. The angular positions of link 1 and 2 \((q_1 \text{ and } q_2, \text{ respectively; } q_1 \text{ is the angle between link 1 and the horizontal axis, and } q_2 \text{ the angle between the links})\) are measured using two high-resolution optical encoders. The design gives both links full 360° of rotational motion. A detailed description of the nonlinear model is given in [14] and [16].

The equation of motion for the Pendubot can be formulated as a nonlinear state-space description

\[
\dot{x}(t) = f(x) + g(x)u(t) = f(x(t), u(t), t),
\]

where \(x = [q_1 \ \dot{q}_1 \ q_2 \ \dot{q}_2]^T\) is the state vector and \(u(t) = [\tau_1 \ 0]^T\) is the input vector \((\tau_1 \text{ is the torque applied to link 1 and } \dot{q}_1, \dot{q}_2 \text{ represent the time derivative of the respective angles})\). We select as output \(y(t) = q_2\). Linearization of this nonlinear model for a specific equilibrium point can be obtained using Taylor series, hence the respective linearization is given as \(\dot{x}(t) = Ax(t) + Bu(t)\), where

\[
A := \frac{\partial f(x(t), u(t), t)}{\partial x} \bigg|_{x^0, u^0},
\]

\[
B := \frac{\partial f(x(t), u(t), t)}{\partial u} \bigg|_{x^0, u^0},
\]

and

\[
x^0 = [q_1^0 \ \dot{q}_1^0 \ q_2^0 \ \dot{q}_2^0]^T; \quad u^0 = [\dot{x}_1^0 \ 0]^T
\]

are the values of \(x\) and \(u\) for the specific equilibrium point. For the Pendubot every equilibrium point must satisfy \(q_2 + q_1 = 90°\).

B. The Experimental Application

In this section, we discuss real-time implementation of the proposed algorithm to the Pendubot. The parameter values of Pendubot model used are given in the Appendix. The objective is to force the angular position of link 2 \((q_2)\) to follow a sinusoidal signal of 70° amplitude. In order to track this signal, it is required to fulfill \(q_1 + q_2 = 90°\) at every trajectory point.

The control goal is to track a sinusoidal signal; in order to obtain it, for each equilibrium point an additional constant signal (offset) has to be added to the sinusoidal signal. The reference to be generated is \(y_r = u_{ofs} + k \sin\alpha t, i = 1, \ldots, 5\), where \(u_{ofs}\) is the required offset for each equilibrium point \((u_{ofs} = q_2^0)\), \(k\) is the amplitude of the sinusoidal signal and \(\alpha\) is its angular frequency.

**Fuzzy Plant:** To model the Pendubot, we propose the five fuzzy sets presented in Fig. 1, with: **BP:** Big Positive; **MP:** Medium Positive; **Z:** Zero; **MN:** Medium Negative; **BM:** Big Negative. The nonlinear model of the Pendubot is linearized around the following equilibrium points \(x_i^0 = [q_{1i}^0 \ \dot{q}_{1i}^0 \ q_{2i}^0 \ \dot{q}_{2i}^0]^T\), \(i = 1, \ldots, 5\), where \(q_{11}^0 = 70°, q_{12}^0 = 35°, q_{13}^0 = 0°, q_{14}^0 = -35°, q_{15}^0 = -70°\) and \(q_{2i}^0 = 90° - q_{1i}^0, i = 1, \ldots, 5\).

For each equilibrium point we obtain a linear system \((A_i, B_i), i = 1, \ldots, 5\). The values of \((A_i, B_i)\) are giving in the Appendix. The output \(y\) is chosen always as \(q_2\). For the Pendubot, the exosystem for each linear region is given as

\[
\dot{w}(t) = Sw(t); \quad w(0) = w_0; \quad y_r(t) = Q_iw(t)
\]

where

\[
w^T = [w_1(t) \ w_2(t) \ w_3(t)]; \quad w_0^T = [1 \ 0 \ 1]
\]

\[
S = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \alpha \\
0 & -\alpha & 0
\end{bmatrix} \quad \forall i = 1, \ldots, 5
\]

\[
Q_i = [-u_{ofs}; -k \ 0]; \quad i = 1, \ldots, 5.
\]
In this paper, each $K_i$ is selected to have the eigenvalues of 
$(A_i + B_i K_i)$ located at $[-9.47 - 9.24 - 5.46 - 5.42]$. This 
pole placement selection presents adequate performance. The 
local control signals are obtained from (4); particular values of 
$K_i$ and $L_i$ for each local controller are given in the Appendix. 
Finally the control signal applied to the systems is given by (7).

**Fuzzy Rules:** There are five fuzzy rules for the plant and for 
the controller. We illustrate only the first one; the other ones are 
included in [17].

1st Plant Rule:

$$x(t) = A_1 x(t) + B_2 u(t)$$

1st Controller Rule:

$$u(t) = C x(t) + Q_2 w(t)$$

C. Experimental Results

The schematic diagram of our experimental setup is shown in 
Fig. 2. The motor is a high-torque 90 VDC permanent magnet 
one. Link positions are measured with BEI 1024 optical en-
coders and a servo amplifier 25A8PWM is used to drive the 
motor, which is operated in torque mode. The DAC card pro-
vides analogical signals between $-10$ and $+10$ V. Our control 
algorithm is implemented in C language, on a Pentium IBM PC 
75 MHz. The sampling time is selected equal to 5 ms.

Performance of the proposed algorithm is illustrated in 
Figs. 3 and 4. Fig. 3 shows results when the sinusoidal signal to 
be tracked is $70 \sin 0.05 t$. In this case, the closed-loop system 
presents good tracking, with no saturation of the control signals. 
Fig. 4 shows also acceptable performances for $70 \sin 0.5 t$. 
Finally, in Fig. 5 we present a Pendubot pictures sequence 
illustrating real-time trajectory tracking for the sinusoidal 
reference $70 \sin 0.05 t$.

In [17] comparisons with results using different techniques 
are discussed. However, due to space restrictions, here we 
include only a brief description. When standard fixed linear 
regulators were implemented, we were unable to go further 
than $50^{\circ}$. Regarding classical gain scheduling algorithms, it 
was impossible to keep the Pendubot stable at the extreme point 
for sinusoidal trajectories with $70^{\circ}$ amplitude. As mentioned 
on the introduction, to the best of our knowledge, all of very 
few published fuzzy scheme for nonlinear system trajectory 
tracking are based on T–S approach [6], [10]–[13]. Taking into 
account this fact, we expect the respective experimental results 
to be similar to ours.

Using the discussed T–S fuzzy controller (5), (6), we were 
able to track a sinusoidal with an amplitude much bigger 
than in [16]. It is also worth mentioning that this controller is 
easy to synthesize due to the use of well-known linear control 
techniques.

1 This reference is available, via e-mail, from the authors.
IV. CONCLUSION

We have presented a T–S fuzzy scheme for trajectory tracking of underactuated mechanism. In particular, we discussed its real-time application to the Pendubot. The obtained results are quite encouraging; it was possible to track sinusoidal signals of big amplitude. The synthesis of this scheme is simple, because it is developed on the basis of well-known linear control methodologies. Research is being pursued in order to solve the problem of Pendubot trajectories crossing singular points.
APPENDIX

Pendubot Parameters

Link 1 mass includes encoder 2 mass; \( l_1 \) and \( l_2 \) are link 1 and link 2 lengths; \( l_{1c} \) and \( l_{2c} \) are the distances to the center of mass of the respective links; \( m_1 \) and \( m_2 \) are the respective mass; \( I_{zz1} \) and \( I_{zz2} \) are the inerces and \( \mu_1 \) and \( \mu_2 \) are friction constants.

\[
\begin{align*}
m_1 &= 0.8293 \text{ Kg} & l_1 &= 0.2032 \text{ m} & I_{zz1} &= 0.005 \text{ Kg-m}^2 \\
m_2 &= 0.3402 \text{ Kg} & l_2 &= 0.3841 \text{ m} & I_{zz2} &= 0.0043 \text{ Kg-m}^2 \\
\mu_1 &= 0.00545 \text{ Kg/s} & l_{1c} &= 0.1551 \text{ m} \\
\mu_2 &= 0.00047 \text{ Kg/s} & l_{2c} &= 0.1635 \text{ m}
\end{align*}
\]

Linear Models

Linear Model for \([20^\circ\ 70^\circ\ 0\ 0]\)

\[
A_1 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
13.032 & -4.0543 & -0.140 & 0.0155 \\
23.931 & 45.948 & 0.18 & -0.055
\end{bmatrix} ;
\]

\[
B_1 = \begin{bmatrix}
0 \\
25.752 \\
-33.182
\end{bmatrix} ;
\quad C_1 = C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T .
\]

Linear Model for \([55^\circ\ 35^\circ\ 0\ 0]\)

\[
A_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
36.127 & -11.2094 & -0.1624 & 0.0236 \\
-20.367 & 59.73 & 0.2746 & -0.075
\end{bmatrix} ;
\]

\[
B_2 = \begin{bmatrix}
0 \\
29.807 \\
-50.403
\end{bmatrix} ;
\quad C_2 = C .
\]

Linear Model for \([90^\circ\ 0^\circ\ 0\ 0]\)

\[
A_3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
48.653 & -15.136 & -0.179 & 0.0284 \\
-48.969 & 68.6277 & 0.3303 & -0.0875
\end{bmatrix} ;
\]

\[
B_3 = \begin{bmatrix}
0 \\
32.881 \\
-60.618
\end{bmatrix} ;
\quad C_3 = C .
\]

Linear Model for \([125^\circ\ -35^\circ\ 0\ 0]\)

\[
A_4 = A_2 ; \quad B_4 = B_2 ; \quad C_4 = C .
\]

Linear Model for \([100^\circ\ -70^\circ\ 0\ 0]\)

\[
A_5 = A_1 ; \quad B_5 = B_1 ; \quad C_5 = C .
\]

Local Control Gains

Linear Gains for \(70 \sin 0.05\bar{\pi}\):

Linear Model 1: \( K_1 = [60.56\ 58.74\ 10.64\ 9.15] \);

\[
L_1 = [-3.02\ 3.02\ 0.08] ;
\]

Linear Model 2: \( K_2 = [21.82\ 21.27\ 4.53\ 3.26] \);

\[
L_2 = [-1.3\ 2.61\ 0.07] ;
\]

Linear Model 3: \( K_3 = [15.95\ 15.95\ 3.56\ 2.41] \);

\[
L_3 = [0\ 2.37\ 0.069] ;
\]

Linear Model 4: \( K_4 = [21.82\ 21.27\ 4.53\ 3.26] \);

\[
L_4 = [1.3\ 2.61\ 0.07] ;
\]

Linear Model 5: \( K_5 = [60.56\ 58.74\ 10.64\ 9.15] \);

\[
L_5 = [3.02\ 3.02\ 0.08] .
\]

Linear Control Gains for \(70 \sin 0.5\bar{\pi}\):

Gains \( K_i \) (\( i = 1, \ldots, 5 \)) are the same as above.

\[
L_1 = [-3.02\ 2.91\ 0.867] ; \quad L_2 = [-1.3\ 2.50\ 0.748] ;
\]

\[
L_3 = [0\ 2.27\ 0.677] ; \quad L_4 = [1.3\ 2.50\ 0.748] ;
\]

\[
L_5 = [3.02\ 2.91\ 0.867] .
\]

ACKNOWLEDGMENT

The authors thank the reviewers for helping to improve the quality of the paper.

REFERENCES


Ofelia Begovich was born in Mexico City, Mexico. She received the B.S.E.E. degree in electronics engineering from the National Polytechnic Institute, Mexico City, in 1983; the M.Sc. degree in electrical engineering from CINVESTAV-IPN (Advanced Studies and Research Center of the National Polytechnic Institute), Mexico City, in 1987, and the Ph.D. degree in electrical engineering from the University of Rennes I, Rennes, France, in 1992. Since 1992, she has been a Professor of Electrical Engineering of graduate program in CINVESTAV-IPN, Guadalajara, Mexico. Her research interests include fuzzy logic and discrete-event systems and their applications to automatic control.

Marcos Maldonado was born in Morelia, Michoacan, Mexico. He received the B.S.E.E. degree in electronics engineering from the Technological Institute of Morelia in 1997 and the M.Sc. degree in electrical engineering from CINVESTAV-IPN (Advanced Studies and Research Center of the National Polytechnic Institute), Guadalajara, Mexico, in 1999. Since then he has worked for different manufacturing companies in Monterrey, Nuevo Leon, Mexico.