\( \varepsilon \)-Equilibrium in LQ differential games with bounded uncertain disturbances: robustness of standard strategies and new strategies with adaptation

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A finite time multi-persons linear-quadratic differential game (LQDG) with bounded disturbances and uncertainties is considered. When players cannot measure these disturbances and uncertainties, the standard feedback Nash strategies are shown to yield to an \( \varepsilon \)-(or quasi) Nash equilibrium depending on an uncertainty upper bound that confirms the robustness property of such standard strategies. In the case of periodic disturbances, another concept, namely adaptive concept, is suggested. It is defined an “adaptation period” where all participants apply the standard feedback Nash strategies with the, so-called, “shifting signal” generated only by a known external exciting signal. Then, during the adaptation, the readjustment (or correction) of the control strategies is realized to estimate the effect of unknown periodic disturbances by the corresponding correction of the shifting vector. After that adaptation period, the complete standard strategies including the recalculated shifting signal are activated allowing the achievement of pure \((\varepsilon = 0)\) Nash equilibrium for the rest of the game. A numerical example dealing with a two participants game shows that the cost functional for each player achieves better values when the adaptive approach is applied.

1. Introduction

During the last decades, a significant interest in the application of some modern concepts in linear-quadratic differential games (LQDG) has increased (see Amato et al. 2002, Olsder 2003, van den Broek et al. 2003). They deal with the optimization of multiple control agents (players) dynamics when none of them can control the decision making of others and each dynamic is affected by the consequences of these decisions. To resolve such a conflicting situation, the concept of Nash equilibrium (or, in general case, quasi-Nash equilibrium) is applied (Tanaka and Yokoyama 1991). A special interest within the last publications is related with the analysis of different uncertainty effects to behavior of players’. In (Basar and Olsder 1982, James and Baras 2000, Amato et al. 2002, van den Broek et al. 2003) linear-quadratic (LQ) games the uncertainty scenarios have been considered using the \( \mathcal{H}_\infty \) approach leading to the min–max formulation where max is taken over the external perturbations of a bounded energy and min with respect to the admissible strategies. In some sense, such min–max player’s strategies can be considered as robust (van den Broek et al. 2003, Engwerda et al. 2005). Any robust control is “passive” by its definition since it may be designed before starting a control (in this case, game) process, or in other words, without the use of current (or, online) information on realized dynamics. There exists another concept of control using this current information. It is related to the designing of really active control strategies which use some sort of “estimates” (direct and indirect) of existing uncertainties (or disturbances). In some sense this is the “adaptive control” concept (Kim and Bai 1981, Engwerda 1991, Sheppard 1998, Khou 2003). Here we follow this second concept.
One of the main contributions of the paper is to define the $\varepsilon$-Nash equilibrium for a differential game and to apply this definition to LQ games with uncertainties. In this paper, we assume the presence of noises (or external perturbation) which have an influence to the dynamics of the players and are not available (measurable) neither a priori nor online. In the first part of the paper, we study the effect of the application of the standard feedback Nash strategies (designed for the game without any unknown disturbances) in noise situations. We show that they provide the so-called, near-Nash (or, $\varepsilon$-Nash) equilibrium (Tanaka and Yokoyama 1991) where the $\varepsilon$-level turns out to be a function of the upper bound for the energy of external noises (perturbations). In some sense, this result demonstrates the “robustness” of the standard game strategies with respect to external uncertainties. As we already mentioned before, such strategies represent the, so-called, passive control actions.

The second contribution of this work is related to the designing of active strategies that include some procedures which use adaptation to external uncertainties. Here, we suppose that the external noise (or perturbation) is bounded and is periodic with an a priori known period. It is well known (Basar and Olsder 1982) that the exact realization of the Nash equilibrium strategies demands the solution of N-coupled matrix Riccati differential equations backward in time together with the coupled vector differential equation (also backward in time) known as the “shifting equation”. In some sense, this second vector equation realizes the “compensation” of external perturbations. But the main trouble in the solution of this equation is that the players need the complete (from the beginning up to the end) information on these perturbations to have a possibility to use it in backward time. Therefore, we consider here only periodical perturbations (with known periods) and propose the active two-stage control strategies for each of the participants: the first stage, called “the adaptation period”, is suggested to reach the period of time where the complete information on the uncertainty is in the game during the robust regime application and the second stage is suggested to be the complete standard control with the exact recalculation of the shifting vector that provides a pure Nash equilibrium. The recalculation of the shifting vector can be made because of the properties of the fundamental matrix of the linear differential equations and the integrations by parts. The corresponding adaptation (or learning) period is exactly the period of unknown external perturbation. As a result, the players are able to estimate the effect of any periodic uncertain perturbation. Once the players have the complete shifting signal, they have the complete optimal backward gain matrices (until the end of the game), moving forward starting from their initial points, or in other words, this permits to estimate exactly the influence of a periodic noise and to compensate it starting from the period of this perturbation.

This paper is organized as follows: §2 represents the model description and the basic assumption accepted in the paper. Section 3 presents the analysis of the standard game strategies (designed for no noise situation) being applied to the linear dynamics with external unknown perturbations. Here it is shown that this strategy is robust providing a $\varepsilon$-Nash equilibrium for the game. In §4 details of the adaptive approach is presented. In §5, a numerical example, comparing the cases when the adaptation strategy is applied and when is not applied, is given, and §6 concludes this study.

2. Players’ model description and basic assumptions

Let us consider an LQDG where the players’ dynamics are covered by linear ordinary differential equations (ODEs)

$$
\dot{x}(t) = A(t)x(t) + \sum_{j=1}^{N} B^j(t)u^j(t) + d(t) + \tilde{\xi}(t)
$$

$$
x(0) = x_0, \quad u^j(t) \in \mathbb{R}^{m_j}, \quad t \in [0, T], \quad T < \infty
$$

with a quadratic cost functional as an individual aim performance

$$
J^j_u(u^j, u') = x^T(T)Q^j_j x(T) + \int_{0}^{T} \left( x^T Q(t) x + \sum_{j=1}^{N} u'^j R^j(t) u^j \right) dt
$$

Here $j$ denotes the number of players ($j = 1, N$), $A(t) \in \mathbb{R}^{n \times n}$ and $B^j(t) \in \mathbb{R}^{m_j \times n}$ are the known system matrices, and $d(t) \in \mathbb{R}^{n}$ is the known exciting signal, $\tilde{\xi}(t) \in \mathbb{R}^{r}$ is the unknown bounded noise, and $x(t) \in \mathbb{R}^{n}$ is the state vector of the game, and $u^j(t) \in \mathbb{R}^{m_j}$ being the control strategies of each j-player.

The performance index $J^j_u(u^j, u')$ (2) of each i-player is given in the standard Bolza form where $u'$ is the strategy for i player and $u'$ are the strategies for the rest of the players (i is the counter-coalition collection of players counteracting to the player with the index i). We will assume also that

$$
Q^j(t) = Q^j T(t) \geq 0, \quad Q^j(t) = Q^j T(t) \geq 0
$$

$$
R^j(t) = R^j > 0, \quad R^j(t) = R^j T(t) \geq 0 \quad (j \neq i)
$$

(3)
2.1 Basic assumptions and definitions

The definitions and assumptions below will be applied in the first part of this paper.

A1 The matrices $A(t), B(t)$ of the game as well as the exciting input $d(t)$ are assumed to be known and be integrable on $[0, T]$ for every participant, i.e., $d(t) \in L^1(0, T; \mathbb{R}^n)$, and $\xi(t)$ is an unknown bounded noises, i.e., for all $t \geq 0$

$$\|\xi(t)\|^2 \leq \beta < \infty$$  

A2 Suppose the class $U^i_{\text{admis}}$ of admissible control actions $u^i(x, t)$ $(i = 1, \ldots, N)$ contains all nonstationary feedback controls satisfying the uniform (on $t$) Lipschitz condition on $x$, for any $t \geq 0$ and any $x, x' \in \mathbb{R}^n$

$$\|u^i(x, t) - u^i(x', t)\| \leq \delta \|x - x'\|$$

Additionally, all admissible controls are assumed to be quadratically integrable in $t$ (as well as the corresponding dynamics $x(t)$) within the time interval $[0, T]$ for any $x$, that is, for any $x \in \mathbb{R}^n$

$$u^i(x', \cdot) \in L^2(0, T; \mathbb{R}^m) \quad \text{and} \quad x(t) \in L^2(0, T; \mathbb{R}^n)$$

2.2 Memoryless feedback Nash equilibrium

2.2.1 $\varepsilon$-Nash equilibrium

Definition 1: The players’ control strategies $u^i(\cdot)$ $(i = 1, \ldots, N)$ are said to be in $\varepsilon$-Nash equilibrium if for any other admissible control strategy $\tilde{u}(\cdot)$ $(i = 1, \ldots, N)$ the following inequalities hold:

$$J^i_T := J^i_T(u^*, \tilde{u}^*) \leq \inf_{u(\cdot) \in U^i_{\text{admis}}} J^i_T(u^i, \tilde{u}^i) + \varepsilon_T^i, \quad i = 1, \ldots, N$$

where $\varepsilon_T^i \geq 0$ is a numerical level characterizing how far the control $u^i(\cdot)$ $(i = 1, \ldots, N)$ deviates from the pure Nash equilibrium ($\varepsilon_T^i = 0$).

The main contribution of the paper is to define the $\varepsilon$-Nash equilibrium for a differential game and to apply this definition to LQ games with uncertainties. To clarify the equation (7), let us consider the illustrative figures 1 and 2. One can see that in case of the pure ($\varepsilon = 0$) Nash equilibrium (if the cost function is convex with respect to the first argument) the minimizer $u^i$ of the functional $J^i_T(u^i, \tilde{u}^i)$ for the $i$th player exactly satisfies (7) with $\varepsilon_T^i = 0$ (see figure 1).

In the case $\varepsilon > 0$ (see figure 2) there exists a set $U^i_{\varepsilon}$ of strategies $(u^i)'$, $(u^i)''$ etc. which satisfies (7) and, hence, each of them can be considered as an $\varepsilon$-Nash equilibrium.

Evidently that, in the presence of immeasurable (but bounded) noise perturbations (or uncertainties) $\xi$, the cost functions $J^i_T(u^i, \tilde{u}^i)$ (which has the same structure as the functional (2) and consider with $J^i_T(u^i, \tilde{u}^i)$ when no noises) turns out to be depend on also on this perturbation $\xi$. Hence, both the cost functionals in the left- and right-hand sides of the inequality below:

$$\tilde{J}^i_T := \tilde{J}^i_T(u^*, \tilde{u}^*) \leq \inf_{u(\cdot) \in U^i_{\text{admis}}} \tilde{J}^i_T(u^i, \tilde{u}^i) + \varepsilon_T^i$$

depends on $\xi$ as well as $\varepsilon_T^i$. The relation between $J^i_T$ and $\tilde{J}^i_T$ is discussed in the Remark 2.

In the case when there are no unknown noises in the game (figure 3), that is, when $\xi(t) \equiv 0$, and under memoryless perfect state information pattern, the following well-known result (see Basar and Olsder 1982) for an $N$ players LQDG (1) and (2) hold: if there exists a set of matrices $P^i(t) \geq 0$, $(i = 1, N)$ satisfying $N$-coupled Riccati differential equations (for the simplicity of the presentation the time dependence of all functions considered below has been omitted) (Abou-Kandil et al. 2003):

$$\dot{P} + \dot{\tilde{P}} + \tilde{A}^T \tilde{P} + \tilde{Q} + \sum_{j=1}^N P^jR^jR^j - R^jR^j - B^jB^j = 0$$

$$P^i(T) = Q^i, \quad i = 1, \ldots, N$$

Figure 1. A pure ($\varepsilon = 0$) Nash equilibrium.

Figure 2. The set $U^i_{\varepsilon}$ of $\varepsilon$-Nash equilibriums.
Remark 1: The equations (9) and (12), are given for known \( d(t) \) (from the end to the beginning of the game in backward time), and works as a "compensator" of the exciting signal \( d(t) \) in optimal strategy (11) for every player.

3. Robustness of classical strategy and \( \epsilon \)-Equilibrium

In the case when \( \tilde{\xi}(t) \neq 0 \), we deal with a noise corrupting the state vector \( x(t) \) and, because of this, the standard Nash equilibrium (with \( \epsilon_T = 0 \)) cannot be insured. Nevertheless, as it is shown below, this standard optimal strategy still becomes to be robust with respect to these unknown perturbations guaranteeing \( \epsilon \)-equilibrium with some \( \epsilon_T \) depending on the upper bound of these perturbations.

In order to make clear the study of robust ness of the standard strategies define the following processes:

- \( x^*(t) \) the players' dynamics when the players use the Nash equilibrium solution (11), (without noises in the game) satisfying

\[
\dot{x}^* = A x^* - \sum_{j=1}^{N} B_j R_j^{-1} B_j^T [P_j x^* + p^j] + d
\]

(this trajectory is used below to evaluate the functional \( J_T(u^*, u^*) \)).

- \( x(t) \) is the players’ dynamics when each player uses the standard strategy (11) under the presence of external noise perturbations, it satisfies

\[
\dot{x} = A x - \sum_{j=1}^{N} B_j R_j^{-1} B_j^T [P_j x + p^j] + d + \tilde{\xi}
\]

To make easy keeping in mind this definition, the corresponding cost functional \( \tilde{J}_T(u^*, u^*) \) is suggested to be referred to as \( \tilde{\tilde{J}}_T(u^*, u^*) \).

- \( \tilde{x}(i) \) is the players’ dynamics when the players use the Nash equilibrium solution except player \( i \) (the index \( i \) reflects this dependence) without noises in the game. It is generated by

\[
\dot{\tilde{x}}(i) = A \tilde{x}(i) + B_i \tilde{u}(\tilde{x})
\]

\[
- \sum_{j \neq i}^{N} B_j R_j^{-1} B_j^T [P_j \tilde{x}(i) + p^j] + d
\]

and the functional is \( \tilde{J}_T(u^*, u^*) \), notice here that \( u(\tilde{x}) \) belongs to \( U_{admis} \).

- \( \tilde{x}(i) \) is the players’ dynamics when every player uses standard strategy except player \( i \) under noises affects, that is, it is governed by

\[
\dot{\tilde{x}}(i) = A \tilde{x}(i) + B_i \tilde{u}(\tilde{x})
\]

\[
- \sum_{j \neq i}^{N} B_j R_j^{-1} B_j^T [P_j \tilde{x}(i) + p^j] + d + \tilde{\xi}
\]

and the corresponding cost functional is referred to as \( \tilde{\tilde{J}}_T(u^*, u^*) \).

As it is shown below, to estimate the cost functionals \( \tilde{\tilde{J}}_T(u^*, u^*) \) which correspond to the \( \epsilon \)-equilibrium formulation (7) it makes sense to compare the value of the last functional \( \tilde{\tilde{J}}_T(u^*, u^*) \) with \( J_T(u^*, u^*) \).

Since the closed-loop system with optimal control \( x^*(t) \) always has a quadratically bounded state vector on \([0, T]\), there exists \( \gamma_T \) such that

\[
\int_0^T \| x^* \|^2 \, dt \leq \gamma_T < \infty
\]

By (6), there exists \( \gamma_{3, T} \) such that

\[
\int_0^T \| \tilde{x} \|^2 \, dt \leq \gamma_{3, T} < \infty
\]
For the details of this statement see the appendix. Define also the following constants:

\[ c_1 := \lambda_1(\gamma_1 + \gamma_2), \quad c_2 := \lambda_2\gamma_1 + \lambda_4\gamma_2 \]
\[ c_3 := \lambda_1(d_1 + d_2) + \lambda_2 V_1(z(0), 0) + \lambda_4 V_2' (\tilde{z}(0), 0), \]
\[ d_1 := \min(L_i)^{-1} |V_1(z(T), T)|, \]
\[ d_2 := \min(L_i)^{-1} |V_2'(\tilde{z}(T), T)| \]

\[ z(t) := x(t) - x^*(t), \quad \tilde{z}(t) = \tilde{x}(t) - \tilde{x}(t) \]

\[ \gamma_1 := \min(L_i)^{-1}\lambda_{\max}(A_i) - 1, \quad \gamma_2 := \min(L_i)^{-1}\lambda_{\max}(A_i)^{-1}, \quad \lambda_1 := \lambda_{\max}(Q_i), \quad \lambda_2 = \sup_{t \in [0, T]} \|Q_i\|, \]
\[ \lambda_3 = 4 \sup_{t \in [0, T]} |Q_i|, \quad \lambda_4 = \sup_{t \in [0, T]} \|Q_i\|, \]
\[ \lambda_5 = 4 \sup_{t \in [0, T]} |Q_i| \sup_{t \in [0, T]} \|M_i\| \]

\[ \bar{Q}^i(P_i) := P_i^T B_i^T (R_i)^{-1} R_i (R_i)^{-1} B_i^T P_i, \]
\[ \bar{Q} := Q - \sum_{j \neq i} \bar{Q}^i(P_i) \]
\[ M_i := \sum_{j = 1}^{N} P_j^T B_j^T (R_j)^{-1} R_j (R_j)^{-1} B_j^T P_j \]

where \( B_i(t), R_i^j(t), B_i^T(t) \), are the known matrices of the system, \( P_i(t) \) and \( P_i'(t) \) are the solutions of (9) and (12). The matrices \( A_i, L_1, L_2 \) are positive definite \( n \times n \) matrices. Here the functions \( V_1(z, t) \) and \( V_2^i(\tilde{z}, t) \) are defined as

\[ V_1(z, t) := z^T S_1(t) z \]
\[ V_2^i(\tilde{z}, t) := \tilde{z}^T S_2^i(t) \tilde{z}, \quad i = 1, \ldots, N \]

where the matrix functions \( S_1(t), S_2^i(t) \) are specified in the theorem below.

**Theorem 1** (the robustness of the standard strategy): Suppose that for the LQ game (1) with the standard optimal strategies \( u^* \) (11) under the assumptions A1 and A2 the following additional supposition holds: there exist positive definite matrices \( \Lambda_1, \Lambda_2^i, \Lambda_3^i, (i = 1, \ldots, N) \)

\[ \tilde{S}_1 + S_1 A + A^T S_1 + L_1 + S_1 \Lambda_1 S_1 = 0, \quad S_1(T) = 0 \]
\[ \tilde{S}_2^i + S_2^i A + A^T S_2^i + L_2^i + S_2^i [\Lambda_2^i + B^T \Lambda_3^i B^T] S_2^i = 0, \quad S_2^i(T) = 0 \]

have positive definite solutions \( S_1 \) and \( S_2^i \), with \( \tilde{A} \) given by (10) and \( A^i \) defined as

\[ \tilde{A}^i = A(t) - \sum_{j \neq i}^{N} B_j^i(t) (R_j^0(t))^{-1} B_j^T(t) P_j(t) \]

Then these standard optimal strategies (11) provide the \( \varepsilon \)-Nash equilibrium of the game, that is, for any admissible \( u' \), satisfying (5) the inequality

\[ \hat{J}_{\varepsilon}^i := \hat{J}_{\varepsilon}^i(u^*, u^*) \leq \hat{J}_{\varepsilon}^i(u', u^*) + \varepsilon \]

with

\[ \varepsilon := c_1 \beta + 2(\gamma_1 T + d_1)^{1/2} + 2(\gamma_2 T + d_2)^{1/2} + c_2 \beta T + \lambda_3 (\gamma_1 T + V_1(0))^{1/2} + \lambda_5 (\gamma_1 T + V_1(0))^{1/2} + c_3 \]

may be guaranteed where \( \hat{J}_{\varepsilon}^i \) denotes the functional obtained by \( u^* \) (11) application affected by unknown perturbations and \( u' \) is any admissible control.

The proof of this theorem and all other statements are given in the appendix.

**Corollary 1**: The values \( \varepsilon \) (28) are consistent in the sense that when uncertainty is absent, i.e., \( \tilde{x}(t) \equiv 0 \), it follows that \( \varepsilon = 0 \) \( (i = 1, \ldots, N) \). Indeed, in this case \( \beta = 0 \) and

\[ z(t) = x(t) - x^*(t) = 0, \quad \tilde{z}(t) = \tilde{x}(t) - \tilde{x}(t) = 0 \]

that implies

\[ V_1(z(t), t) = V_2^i(\tilde{z}(t), t) \equiv 0 \]

and, hence,

\[ d_1 = 0, \quad d_2 = 0 \quad (i = 1, \ldots, N), \quad c_3 = 0 \]

\[ \tilde{\Lambda}_1, \tilde{\Lambda}_2^i, \tilde{\Lambda}_3^i \] The normalizing matrices are introduced here to have the possibility to work with components of different physical nature, for example, \([x_1] = \text{kg}, [x_2] = \text{cm}, [x_3] = \text{dollar} \) and etc. and make them “normalized” or independent of any physical dimension.
Corollary 2: For average cost LQDGs with dynamics given by (1) and the average cost functionals defined as

\[ J_{av, T}(u', u^*) := T^{-1}^J_T(u', u^*) \]

and when there exist the limits

\[ \limsup_{T \to \infty} T^{-1} \gamma_T := p < \infty, \quad \limsup_{T \to \infty} T^{-1} \gamma_{3, T} := q < \infty \]

the standard optimal strategies \( u^* \) (11) for any large enough \( T \) provide the \( \varepsilon \)-Nash equilibrium of the game, that is, for any admissible \( u' \), satisfying (5) the inequalities (\( i = 1, \ldots, N \))

\[ \tilde{J}_{av, T}(u'^*, u^*) \leq \tilde{J}_{av, T}(u', u^*) + \varepsilon^T_{av, T} \]

with \( \varepsilon^T_{av, T} \) equal to

\[ \varepsilon^T_{av, T} = T^{-1} \varepsilon_T = \varepsilon_2 \beta + (\lambda_2 \sqrt{p \gamma_1} + \lambda_5 \sqrt{q \gamma_2}) \sqrt{\beta} + O \left( \frac{1}{\sqrt{T}} \right) \]

Remark 2: It is easy to notice that the definition (8) can be rewritten in the following equivalent form:

\[ \tilde{J}_{T}(u'^*, u^*) \leq \inf_{u' \in U_{adm}} \tilde{J}_{T}(u', u^*) + \varepsilon_T = \tilde{J}_{T}(u'^*, u^*) + \varepsilon_T \]

\[ + \left( \inf_{u' \in U_{adm}} \tilde{J}_{T}(u', u^*) - \tilde{J}_{T}(u'^*, u^*) + \varepsilon_T \right) \]

\[ \leq J_T(u'^*, u^*) + \varepsilon_T \]

where

\[ \varepsilon_T := \varepsilon_T + \left| \inf_{u' \in U_{adm}} \tilde{J}_{T}(u', u^*) - \tilde{J}_{T}(u'^*, u^*) \right| \]

that gives the comparisons of the cost functionals in the presence of uncertainties (when of the \( \varepsilon \)-Nash equilibrium strategies is applied) with the same cost functionals corresponding to the dynamics without noise (or uncertainties). Sure, the modified \( \varepsilon_T \) can be estimated by the similar calculations as \( \varepsilon_T \).

4. Adaptive case: readjustment of the shifting vector

One possible way of enhancing the robustness of the equilibrium in an LQ game is to estimate the “discrepancies” between the applied game model (which does not contain any signal associated with uncertainty) and the actual game (usually affected by uncertainties or noises) with the following incorporation of this information in the control strategies of the players in a proper way, or, in other words, adapt the control strategies. In LQ case, the shifting vector plays an important role since the information of the uncertainty is introduced to the control strategies for all the players exactly through this equation. This makes it possible to overcome (as we already said before) the problems related to these uncertainties. So, for this case, the key point is to obtain the complete shifting vector which incorporates the information on unknown signal without the direct uncertainty recovering. In this work, we prove that this equation can be reevaluated to attain the complete standard strategies including the recalculated shifting signal after some period, allowing the players to achieve the exact Nash equilibrium only with the information available during the game. Sure, that such reevaluation cannot be done for all possible perturbations. That is why an additional assumption on uncertainty properties is required, namely, below and we suppose that the uncertain signal is periodic with a priori known period.

4.1 Additional assumption

A3 Considering \( \tilde{\xi}(t) \) as an unknown perturbation in (1), let us assume that it is periodical, with a known period \( \tau \in [0, T/2) \), that is,

\[ \tilde{\xi}(t) = \tilde{\xi}(t + n \tau), \quad n = 1, 2, \ldots, \quad \forall t \in [0, \tau], \]

\[ T > 2 \tau \]

Define an “adaptation period” \( t_{adap} \in [\tau, T] \) as a time required for each player to recuperate exactly the corresponding shifting vector. Hence, \( t_{adap} \) is a time starting from which we are able to design exactly the shifting equation.

Define the joint excitation vector \( \tilde{d}(t) \) as

\[ \tilde{d}(t) = d(t) + \tilde{\xi}(t) \]

This vector \( \tilde{d}(t) \) participates in the formation of the optimal shifting vector \( p(t) \) (12), namely,

\[ p^j + \tilde{A}^T p^j + \sum_{j=1}^{N} \mathcal{F}^j_0 p^j + \eta^j \tilde{d} = 0, \quad p^j(T) = 0 \]

\[ \mathcal{F}^j := P^j B^i R^j_0 R^j_0^{-1} B^i + R^j_0^{-1} B^i \tilde{d} \]

or, in the extended form:

\[ \dot{p} = -[A^T + F]p + P \tilde{d}; \quad p(T) = 0 \]
where the extended vector and matrices and defined as:

\[
P = \begin{bmatrix}
p^i \\
\vdots \\
p^N
\end{bmatrix}, \quad A = \begin{bmatrix}
\hat{A} & \cdots & \Theta \\
\vdots & \ddots & \vdots \\
\Theta & \cdots & \hat{A}
\end{bmatrix}, \\
F = \begin{bmatrix}
\mathcal{F}^{11} & \cdots & \mathcal{F}^{1N} \\
\vdots & \ddots & \vdots \\
\mathcal{F}^{N1} & \cdots & \mathcal{F}^{NN}
\end{bmatrix}
\]

\( A \in \mathbb{R}^{nN \times nN}; \quad p \in \mathbb{R}^{nN}; \quad F \in \mathbb{R}^{nN}; \quad \Phi \in \mathbb{R}^{nN \times nN} \)

By Cauchy formula applied to (34) one has

\[
p(t) = \int_{s=0}^{t} \Phi(t,s)p(s)\tilde{d}(s)ds
\]

where the transition matrix function \( \Phi(t,s) \) satisfies the following ODEs:

\[
\begin{align*}
\frac{\partial}{\partial t} \Phi(t,s) &= -[A^T + F]\Phi(t,s) \\
\frac{\partial}{\partial s} \Phi(t,s) &= \Phi(t,s)[A^T + F] \\
\Phi(t,t) &= I
\end{align*}
\]

For any admissible strategy \( u(\cdot) = [u^1(\cdot) \cdots u^n(\cdot)]^T \) the game dynamics may be expressed as

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + S(t)u(t) + \tilde{d}(t) \\
S &= [B^1 \cdots B^n]; \quad u = [u^1 \cdots u^n]^T
\end{align*}
\]

From (37) it follows that \( \tilde{d}(t) \) (33) can be represented as

\[
\tilde{d}(t) = \dot{x}(t) - A(t)x(t) - S(t)u(t)
\]

Substitution (38) in (35) implies

\[
p(t) = \int_{s=0}^{t} \Phi(t,s)p(s)[\dot{x}(s) - A(s)x(s) - S(s)u(s)]ds
\]

Integrating by parts the first term in the right-hand side, containing the derivative, leads to the following representation

\[
\begin{align*}
\int_{s=0}^{t} \Phi(t,s)p(s)x(s)ds &= \Phi(t,\tau)Q_x(x(\tau)) - P(t)x(t) \\
- \int_{s=0}^{t} \frac{\partial \Phi(t,s)}{\partial s}p(s)x(s)ds \\
- \int_{s=0}^{t} \Phi(t,s)\frac{\partial p(s)}{\partial s}x(s)ds &= \Phi(t,\tau)Q_x(x(\tau)) - P(t)x(t) \\
- \int_{s=0}^{t} \Phi(t,s)[A^T + F]p(s)x(s)ds - \int_{s=0}^{t} \Phi(t,s)\frac{\partial p(s)}{\partial s}x(s)ds
\end{align*}
\]

where \( \partial p(s)/\partial s \) is defined by (25). Substitution (40) into (39) finally implies

\[
p(t) = \Phi(t,\tau)Q_x(x(\tau)) - P(t)x(t) - \int_{s=0}^{t} \Phi(t,s)\frac{\partial p(s)}{\partial s}x(s)ds \\
- \int_{s=0}^{t} \Phi(t,s)[A^T + F]p(s)x(s) + P(s)x(s) \\
\times [A(s)x(s) + S(s)u(s)]ds
\]

or, in a more simple form,

\[
p(t) = \Phi(t,\tau)Q_x(x(\tau)) - P(t)x(t) - \int_{s=0}^{t} \Phi(t,s)\Psi(s)ds
\]

Defining

\[
\tilde{p}(t) := \int_{s=0}^{t} \Phi(t,s)\Psi(s)ds
\]

which can be represented equivalently in the differential form as

\[
\dot{p}(t) = -[A^T + F] - \Psi(t), \quad \tilde{p}(t) = 0
\]

finally, we obtain:

\[
p(t) = \Phi(t,\tau)Q_x(x(\tau)) - P(t)x(t) - \tilde{p}(t)
\]

where \( \tilde{p}(t) \) satisfies (42).

**Remark 3:** As it follows from (35), the shifting vector \( p(t) \) is a “function” of unknown perturbation \( \xi(t) \) (which is periodic now). But by (38), the joint unknown perturbation \( \tilde{d}(t) \), which includes \( \dot{\xi}(t) \), can be exactly presented in terms of an applied admissible control \( u(s) \) \( s \in [0, \tau] \) and the corresponding dynamics \( x(t), \dot{x}(t) \). Fortunately, the state derivative \( \dot{x}(t) \) can be excluded from the presentation (39) by the simple integration by parts.

**Remark 4:** Formula (41) realizes the optimal shifting vector (and, hence, the complete optimal Nash strategies) for all the players. Notice that the differential form of this expression (34) is not realizable since the signal \( \tilde{d}(t) \) contains the unknown periodic disturbance \( \xi(t) \) making impossible its realization. Contrarily, the integral form (41) of the optimal shifting vector as well as the forms (42) and (43) are perfectly realizable since it includes only the measurable data \( [A(s)x(s) + S(s)u(s)] (s \in [0, \tau]) \) available during the first (adaptation) period.
5. Numerical example

For the LQ game governed by the equation (1) and by the performance index given in (2) consider the next data corresponding to a time-varying case:

\[
A = \begin{pmatrix}
0 & -2 \\
1 + 0.5 \sin t & -3 \\
\end{pmatrix}, \quad
B_1 = \begin{pmatrix}
1 + 0.1 \cos t \\
0 \\
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
1 + 0.03 \exp^{-t} \\
1 \\
\end{pmatrix}, \quad
\xi(t) = \begin{pmatrix}
15 \sin \pi t \\
20 \sin \pi t \\
\end{pmatrix},
\]

\[
d(t) = \begin{pmatrix}
0.1 \sin t \\
0.2 \sin t \\
\end{pmatrix}
\]

\[T = 10 \text{ sec}, \quad \tau = 2 \text{ sec}\]

\[
R_{11} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \quad
R_{22} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix};
\]

\[
Q_1 = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \quad
Q_2 = \begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix};
\]

\[
Q_1 = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \quad
Q_2 = \begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix};
\]

We present the values of the functionals for three different cases:

1. The functional evaluated from 0 to \(T\) using the adaptation period and readjusting the shifting equation.
2. The functional evaluated from 0 to \(T\) using only the known part of the exiting signal without the adaptation of the shifting equation.
3. The functional evaluated from 0 to \(T\) using the complete information the exiting disturbance signal.

\[
\begin{align*}
\text{Comple Information} & \quad \text{With Adapation} & \quad \text{Without Adaptation} \\
J_1^1 &= 9.0172 \times 10^3 & J_1^1 &= 9.6207 \times 10^3 & J_1^1 &= 1.0341 \times 10^4 \\
J_2^1 &= 1.6227 \times 10^4 & J_2^1 &= 1.7562 \times 10^4 & J_2^1 &= 1.8832 \times 10^4
\end{align*}
\]

Based on the simulation results presented in (44), one can see that

- The obtained \(\varepsilon^i\) characterizing the corresponding robustness of near-Nash equilibrium constitute around 10% of the related cost function values if compare with complete information case.

Figure 3. Plots of the game.
– The use of the adaptation period constitutes around 1% of the related cost function values compared with a complete information case, this means that the value of the cost function most recovers the values of the complete information case.

6. Conclusions

A finite time multi-persons (LQDG) with bounded disturbances and uncertainties is shown to be resolved using two different approaches: the first one is passive, based on the robustness property of the optimal strategies, and the second one is adaptive (in some sense) since it uses the adaptation of the players’ strategies. When players cannot measure these disturbances and uncertainties, the standard feedback Nash strategies are shown to yield to a quasi Nash-equilibrium depending on an uncertainty upper bound that confirms the robustness property of such standard strategies. In the case of periodic disturbances a readjust of the shifting vector was suggested. This recalculation was made exactly thanks to the properties of the ODE’s after the so called “adaptation period” when all participants apply the standard feedback Nash strategies with the, so-called, “shifting signal” generated only by a known external exciting signal. After that period the complete standard strategies with “pre-adaptation” including the recalculated shifting signal were activated. Such strategies permit to estimate exactly the influence of this periodic noise and to compensate it starting from the end of the period of this perturbation making a quasi Nash-equilibrium an exact (pure) Nash-equilibrium (with \( \varepsilon_T = 0 \)). A numerical example, dealing with a two participant game show that the cost functional for each player achieves better values when the adaptive approach is applied.

Appendix

**Proof of Theorem 1:** Since the closed-loop system with optimal control always has a quadratically bounded state vector, there exists \( \gamma_T \) such that

\[
\int_0^T \| x^* \|^2 dt \leq \gamma_T < \infty \tag{45}
\]

The Nash equilibrium definition (7) gives

\[
J_T^p(u^*, u^*) \leq J_T(u^*, u^*)
\]

Adding and subtracting \( \hat{J}_T^p(u^*, u^*) \) and \( \hat{J}_T(u^*, u^*) \) from both sides gives

\[
\begin{align*}
\hat{J}_T^p(u^*, u^*) & \leq \hat{J}_T(u^*, u^*) + \Delta J_{T1} + \Delta J_{T2} \\
\Delta J_{T1} & := \hat{J}_T(u^*, u^*) - J_T^p(u^*, u^*), \\
\Delta J_{T2} & := J_T(u^*, u^*) - \hat{J}_T(u^*, u^*) \\
\end{align*}
\]

In what follows the terms \( \Delta J_{T1} \) and \( \Delta J_{T2} \) will be estimated.

**\( \Delta J_{T1} \)-term estimation.** Under the Nash equilibrium solution \((u^i), i = 1, \ldots, N\), given by (11), the following players’ dynamics will be generated by (15) and the corresponding cost functional is as follows:

\[
J_T^i(u^i, u^*) = x^T(T)Q_{ji}x^*(T)
\]

\[
+ \int_0^T \left\{ x^TQ^i x^* + \sum_{j=1}^N (x^T \tilde{Q}^i(P^j)) x^* - x^T + p^T L \bar{p}^j + 2M \bar{p} \right\} dt
\]

\[
L^i := B(R^i)^{-1} R^i(R^i)^{-1} B^TA^i
\]

The trajectory \( x \), controlled by (11) when noises exist, is given by (16) and the corresponding cost function \( \hat{J}_T(u^i, u^*) \) is

\[
\hat{J}_T(u^i, u^*) = x^T(T)Q_{ji}x(T) + \int_0^T \left\{ x^TQ^i x + \sum_{j=1}^N (x^T \tilde{Q}^i(P^j)) x^* + p^T L \bar{p}^j + 2M \bar{p} \right\} dt
\]

So, \( \Delta J_{T1} \) can be computed as (using (22))

\[
\Delta J_{T1} = x^T(T)Q_{ji}x(T) - x^T(T)Q_{ji}x^*(T)
\]

\[
+ \int_0^T \left\{ x^TQ^i x - x^TQ^i x^* + \sum_{j=1}^N (x^T \tilde{Q}^i(P^j)) x^* - \sum_{j=1}^N (x^T \tilde{Q}^i(P^j)) x^* + 2M \bar{p} \right\} dt \tag{46}
\]

that implies

\[
\Delta J_{T1} = x^T(T)Q_{ji}x(T) - x^T(T)Q_{ji}x^*(T)
\]

\[
+ \int_0^T \left\{ x^T(Q^i + \tilde{Q}^i(P^j)) x - x^T(Q^i + \tilde{Q}^i(P^j)) x^* + 2M \bar{p} \right\} dt = x(T)^T \bar{Q}^i x(T) - x^*(T)^T \bar{Q}^i x^*(T)
\]

\[
+ \int_0^T \left\{ (x - x^*)^T \bar{Q}^i (x + x^*) + 2M \bar{p} \right\} dt
\]
Adding and subtracting the term $x^*$ in the last identity, we derive
\[
(x^T(T) - x^T(T))Q'(x(T) - x^T(T)) + 2x^*(T)
\]
\[
+ \int_0^T \left( (x-x^*)^T \tilde{Q}(x-x^* + 2M^o(T-x^*)) \right) dt
\]
\[
= (x(T) - x^*(T))^T Q'(x(T) - x^*(T)) + 2x^*(T))
\]
\[
+ \int_0^T \left( (x-x^*)^T \tilde{Q}(x-x^* + 4M^o(x-x^*)^T \tilde{Q} x^*) \right) dt
\]
\[
\leq \|x(T) - x^*(T)\|^2_{Q} + 2\|x(T) - x^*(T)\| \|Q'_i x^*\|
\]
\[
+ \int_0^T \left( \|x-x^*\|^2_{\tilde{Q}} + 4M^o\|x-x^*\| \tilde{Q} x^* \right) dt
\]
\[
\leq \lambda_{\max}(\tilde{Q}) (\|x(T) - x^*(T)\|^2 + 2\|x(T) - x^*(T)\| \|x^*\|)
\]
\[
+ \sup_{\tau \in [0,T]} \|\tilde{Q}\| \int_0^T \left( \|x-x^*\|^2 + 4 \sup_{\tau \in [0,T]} \|M^o\| \|x-x^*\| \|x^*\| \right) dt
\]

Using $z$, defined in (23), it follows
\[
\dot{z} = \tilde{A}z + \tilde{\xi}
\]

For the energetic function $V_1$, defined in (23), with $\tilde{A}$ given by (10) the time derivative is
\[
\dot{V}_1 := 2z^T S_1 \dot{z} = 2z^T S_1 (\tilde{A}z + \tilde{\xi}) + z^T \tilde{S}_1 z
\]
\[
= z^T (S_1 \tilde{A} + \tilde{A}^T S_1 + L_1)z - z^T L_1 z + 2z^T S_1 \xi + z^T \tilde{S}_1 z
\]
\[
\leq z^T \left( \tilde{S}_1 + S_1 \tilde{A} + \tilde{A}^T S_1 + L_1 + S_1 L_1 \right) z
\]
\[
- z^T L_1 z + \|\tilde{\xi}\|^2_{(\lambda_i)^{-1}}
\]

where $L_1$ is a positive definite matrix. Select $S_1$ as a solution to the differential Riccati equation (25) and was used the inequality:
\[
X^T Y + Y^T X \leq X^T \Lambda X + Y^T \Lambda^{-1} Y
\]

Integrating $V_1$ implies
\[
\int_0^T \dot{V}_1 dt \leq - \int_0^T z^T L_1 z dt + \int_0^T \|\tilde{\xi}\|_{(\lambda_i)^{-1}}^2 dt
\]

and, hence,
\[
\lambda_{\min}(L_1) \int_0^T \|z\|^2 \leq \int_0^T z^T L_1 z dt
\]
\[
\leq \int_0^T \|\tilde{\xi}\|_{(\lambda_i)^{-1}}^2 dt - V_1(z(T), T) + V_1(0)
\]
\[
\leq \int_0^T \|\tilde{\xi}\|_{(\lambda_i)^{-1}}^2 dt + V_1(0)
\]

Using (22), yields
\[
\lambda_{\min}(L_1)^{-1} \int_0^T \|\tilde{\xi}\|_{(\lambda_i)^{-1}}^2 dt = \lambda_{\min}(L_1)^{-1} \int_0^T \|\tilde{\xi}\|_{(\lambda_i)^{-1}}^2 dt
\]
\[
\leq \lambda_{\min}(L_1)^{-1} \lambda_{\max}(\tilde{A})^{-1} \int_0^T \|\tilde{\xi}\|^2 dt
\]
\[
+ V_1(0) \leq \gamma_1 \int_0^T \beta dt + V_1(0)
\]

that finally leads to
\[
\int_0^T \|z\|^2 dt \leq \gamma_1 \beta T + V_1(0).
\]

Applying the Jensen inequality, and using (22) together with the last inequalities, it follows:
\[
F_T := \lambda_1 \left( \|z(T)\|^2 + 2\sqrt{\|z(T)\|^2 \|x^*(T)\|^2} \right)
\]
\[
+ \lambda_2 \int_0^T \|z\|^2 dt + \lambda_3 \int_0^T \|z\|^2 dt \sqrt{\int_0^T \|x^*\|^2 dt}
\]

Using (47), which is also valid within the interval $(T, T - \Delta)$ for a $\Delta$ small enough, the inequality
\[
\lambda_{\min}(L_1) \|z(T)\|^2 \leq \|\beta\|_{(\Lambda_i)^{-1}}^2 \Delta - V_1(z(T), T)
\]
\[
+ V_1(z(T - \Delta), T - \Delta) + o(\Delta)
\]

is derived. Dividing by $\Delta$ and tending it to zero implies
\[
\|z(T)\|^2 \leq \lambda_{\min}(L_1)^{-1} \left( \|\beta\|_{(\Lambda_i)^{-1}}^2 + \|\dot{V}_1(z(T), T)\| \right) \leq \gamma_1 \beta + d_1
\]

and, hence,
\[
F_T \leq \lambda_1 (\gamma_1 \beta + d_1) + 2\sqrt{\gamma_1 \beta + d_1 \sqrt{T}}
\]
\[
+ \lambda_2 (\gamma_1 \beta T + V_1(0)) + \lambda_3 \sqrt{T} \sqrt{\gamma_1 \beta T + V_1(0)}
\]

(48)

that provides the bound for $\Delta J_{T1}$.

**$\Delta J_{T\Delta}$-term estimation.** When the players use the equilibrium solution (except player $i$) the strategy $u := ((u^T)^i, (u^o)^T)$ generates the following player’s dynamics $\tilde{x}(t|i)$ $(i = 1, \ldots, N)$ (17) (for simplicity the dependence of $\tilde{x}(t|i)$ on $t$ and $i$ has been omitted). The corresponding cost function is evaluated as follows:
\[
J_{T\Delta}^i(u^i, \tilde{u}^o) = \tilde{x}^T(T) Q'_i \tilde{x}(T)
\]
\[
+ \int_0^T \left( \tilde{x}^T Q' \tilde{x} + u^T R^i u \tilde{x}^T \tilde{R}^i u \right)
\]
\[
+ \sum_{j \neq i} \left[ \tilde{x}^T Q^i (P^j) \tilde{x} + p^j T L^j p^j \right] + 2 \tilde{M}^i \tilde{x} dt
\]
When the players use the robust strategies $\hat{u}(t|i)$, $i = 1, \ldots, n$, the trajectories are (18), the corresponding cost functions are

$$
\dot{J}_T'(u', u^*) = \dot{x}^T(T)Q_f\dot{x}(T)
+ \int_0^T \left\{ \dot{x}^T(T)Q_f\dot{x} + u^TRu' + \sum_{j\neq i} \dot{\hat{x}}^T(P_j)\dot{x} + p^{ji} T_x p^j + 2\hat{M}\dot{x} \right\} dt
$$

So, the term of the joint function related to $\Delta J_{T2}$ can be computed as follows:

$$
\Delta J_{T2} = \left[ \dot{x}^T(T)Q_f\dot{x}(T) - \dot{x}^T(T)Q_f\dot{x}(T) \right]
+ \int_0^T \left\{ \dot{x}^T(T)Q_f\dot{x} - \dot{x}^T(T)Q_f\dot{x} \right\} dt
+ \sum_{j\neq i} \left( \dot{\hat{x}}^T(P_j)\dot{x} + x^T(P_j)\dot{x} + 2\hat{M}\dot{x} \right) dt
$$

which implies

$$
\Delta J_{T2} = \dot{x}^T(T)Q_f\dot{x}(T) - \dot{x}^T(T)Q_f\dot{x}(T)
+ \int_0^T \left\{ \dot{x}^T(T)Q_f\dot{x} - \dot{x}^T(T)Q_f\dot{x} \right\} dt
+ \sum_{j\neq i} \left( \dot{\hat{x}}^T(P_j)\dot{x} + x^T(P_j)\dot{x} + 2\hat{M}\dot{x} \right) dt
$$

Using $\hat{Q}_i$, defined in (22), adding and subtracting $\dot{x}$ from both sides of the last equality implies:

$$
\Delta J_{T2} = (\dot{x}(T) - \dot{x}(T))^TQ_f(\dot{x}(T) - \dot{x}(T)) + 2\hat{x}(T)
+ \int_0^T (\dot{x} - \dot{x})^T \hat{Q}_i(\dot{x} - \dot{x} + 2\dot{x}) + 2\hat{M}\dot{x} - \dot{x}) dt
\leq \lambda_{\max}(Q_i)(\|\dot{x}(T) - \dot{x}(T)\|^2)
+ 2\|\dot{x}(T) - \dot{x}(T)\|\|\dot{x}(T)\| + \sup_{\tau \in [0, T]}\left\| \hat{Q}_i \right\|
\times \int_0^T \left\{ \|\dot{x} - \dot{x}\|^2 + 4 \sup_{\tau \in [0, T]}\left\| \hat{M}\|\|\dot{x} - \dot{x}\|\|\dot{x}\| \right\} dt
$$

Using $\ddot{z}(t|i)$, defined in (24)

$$
\ddot{z} = \dot{A}\dot{z} + B'\dot{u}(\dot{x}) - B'u(\dot{x}) - \ddot{\xi}
$$
is derived. The relation (24), the definition (2), and (26) lead to the next inequality:

$$
\dot{V}_2 = 2\dot{z}^T S_2 \ddot{z} + \ddot{z}^T S_2 \ddot{z}
= 2\dot{z}^T S_2 \left( \dot{A}\dot{z} + B'\dot{u}(\dot{x}) - B'u(\dot{x}) - \ddot{\xi} \right) + \ddot{z}^T S_2 \ddot{z}
\leq 2\dot{z}^T S_2 \dot{A}\dot{z} - 2\dot{z}^T S_2 \dot{A}\ddot{z} + \ddot{z}^T S_2 \ddot{z}
+ (\ddot{\xi}, \left[ S_2 B' A_i^{-1} B^T S_2 + \ddot{\lambda}_{\max}(A_i) \right] \ddot{\xi})
\leq \dot{z}^T \left( S_2 + S_2 \dot{A} + A_i^T S_2 + \dot{L}_2 + \dot{S}_2^T \right)
\times \left[ A_i^2 + B_i^2 A_i^{-1} B_i^T \right] \ddot{z}
+ \left\| \dot{\bar{z}} \right\|_{(A_i)}^2
+ \ddot{L}_2 := L_2 + \dddot{\lambda}_{\max}(A_i) > 0
$$

where $S_2$ satisfies (25). Integrating $V_2$ gives

$$
V_2(\ddot{z}(T), T) - V_2(0) \leq -\int_0^T \|\ddot{z}\|^2 dt + \int_0^T \left\| \dot{\bar{z}} \right\|_{(A_i)}^2 dt
$$

and, hence,

$$
\lambda_{\min}(L_2) \int_0^T \|\ddot{z}\|^2 dt \leq \int_0^T \|\ddot{z}\|^2 dt
\leq \int_0^T \left\| \dot{\bar{z}} \right\|_{(A_i)}^2 dt - V_2(\ddot{z}(T), T) + V_2(0)
\leq \int_0^T \left\| \dot{\bar{z}} \right\|_{(A_i)}^2 dt + V_2(0)
\leq \lambda_{\max}(A_i^{-1}) \int_0^T \left\| \dot{\bar{z}} \right\|^2 dt + V_2(0)
$$

Using (22), finally yields

$$
\int_0^T \|\ddot{z}\|^2 dt \leq \gamma_2 \beta T + V_2(0)
$$

It is possible to say that due (5), used in (18), we can ensure the existence of a quadratic bound for $\dot{x}$ that is, exists $\gamma_3$ such that

$$
\int_0^T \|\ddot{x}\|^2 dt \leq \gamma_3 < \infty
$$

Given the inequalities above and applying the Jensen’s inequality, it follows:

$$
\dot{F}_T := \lambda_1 \|\dot{z}(T)\|^2 + 2\sqrt{\|\dot{z}(T)\|^2} \|\dot{z}(T)\|^2
+ \lambda_4 \int_0^T \|\ddot{z}\|^2 dt + \lambda_3 \int_0^T \|\ddot{z}\|^2 dt \int_0^T \|\ddot{z}\|^2 dt.
$$
Using (49) gives

\[
\lambda_{\min}(L_2^I) \| \tilde{z}(T) \|^2 \Delta \\
\leq \| \tilde{\xi} \|^2_{(\Lambda_j)^{-1}} \Delta - V_2^q(T) + V_2^q(T - \Delta) + o(\Delta).
\]

Dividing by \( \Delta \), it follows

\[
\| \tilde{z}(T) \|^2 \leq \lambda_{\min}(L_2^I)^{-1} \left( \| \tilde{\xi} \|^2_{(\Lambda_j)^{-1}} + | V_2^q(T) | \right) \leq \gamma_2 \beta + d_2,
\]

that implies

\[
F_T^2 \leq \lambda_1(\gamma_2 \beta + d_2) + 2\sqrt{\gamma_2 \beta + d_2} \sqrt{\gamma_3(T)} \\
+ \lambda_4(\gamma_2 \beta T + V_2^q(0)) + \lambda_5 \sqrt{V_2^q(N)} \sqrt{\gamma_2 \beta T + V_2^q(0)}.
\]

\[ (50) \]

Summing (48) and (50) proves the theorem.

References


