Kinematics and Diferential Kinematics of Binocular Robot Heads

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Abstract—This paper presents an analysis of the kinematics of a binocular head, and a new formulation of its differential kinematics using the Conformal Geometric Algebra framework. The experimental part includes the smooth visual tracking of 3D objects.

I. INTRODUCTION

When studying the kinematics of mechanisms different frameworks has been employed, e.g. vector calculus, quaternions algebra or linear algebra, being the last one the most used. However, in these frameworks it is very complicated to handle the kinematics and dynamics involving points, lines and planes. In this paper, we show how the mathematical treatment is much easier when we handle it in the conformal geometric algebra framework.

In this paper, we show that using the conformal geometry, it is possible to find a simple formulation for the kinematics of a binocular head. In addition, we show that when we have the kinematics equation, it is possible to formulate the differential kinematics without computing the derivative of such equation.

We show also how to deal with the singulaties introduced by the Jacobian. Finally, we implement a velocity control law for object tracking using a binocular head.

II. GEOMETRIC ALGEBRA: AN OUTLINE

Let \( G_n \) denote the geometric algebra of n-dimensions, this is a graded linear space. As well as vector addition and scalar multiplication we have a non-commutative product which is associative and distributive over addition - this is the geometric or Clifford product. A further distinguishing feature of the algebra is that any vector squares to give a scalar. The geometric product of two vectors \( a \) and \( b \) is written \( ab \) and can be expressed as a sum of its symmetric and antisymmetric parts

\[
ab = a\cdot b + a\wedge b,
\]

where the inner product \( a\cdot b \) and the outer product \( a\wedge b \) are defined by

\[
a \cdot b = \frac{1}{2}(ab + ba)
\]

\[
a \wedge b = \frac{1}{2}(ab - ba).
\]

The inner product of two vectors is the standard scalar or dot product and produces a scalar. The outer or wedge product of two vectors is a new quantity which we call a bivector. We think of a bivector as a oriented area in the plane containing \( a \) and \( b \), formed by sweeping \( a \) along \( b \).

Thus, \( b \wedge a \) will have the opposite orientation making the wedge product anti-commutative as given in equation 2. The outer product is immediately generalizable to higher dimensions - for example, \( (ab)\wedge e \), a trivector, is interpreted as the oriented volume formed by sweeping the area \( a \wedge b \) along vector \( e \). The outer product of \( k \) vectors is a \( k \)-vector or \( k \)-blade, and such a quantity is said to have grade \( k \). A multivector (linear combination of objects of different type) is homogeneous if it contains terms of only a single grade.

A. The Geometric algebra of 3-D space

In this paper we will specify a geometric algebra \( G_n \) of the \( n \) dimensional space by \( G_{p,q,r} \), where \( p, q \) and \( r \) stand for the number of basis vector which squares to 1, -1 and 0 respectively and fulfill \( n = p + q + r \).

We will use \( e_i \) to denote the vector basis \( i \). In a Geometric algebra \( G_{p,q,r} \), the geometric product of two basis vector is defined as

\[
e_i e_j = \begin{cases} 1 & \text{for } i = j, 1,\ldots, p \\
-1 & \text{for } i = j, p + q + r \\
0 & \text{for } i \neq j
\end{cases}
\]

This leads to a basis for the entire algebra:

\[
\{1, e_1, \ldots, e_p, e_{p+1} \ldots, e_{p+q}, e_{p+q+1} \ldots, e_{p+q+r} \}
\]

Any multivector can be expressed in terms of this basis.

In the 3-D space there are multivectors of grade 0 (scalars), grade 1 (vectors), grade 2 (bivectors), grade 3 (trivectors), etc... up to grade \( n \). Any two such multivectors can be multiplied using the geometric product. Consider two multivectors \( A \) and \( B \), of grades \( r \) and \( s \) respectively. The geometric product of \( A \) and \( B \) can be written as

\[
A \cdot B = \langle AB \rangle_r + \langle AB \rangle_{r+s-2} + \cdots + \langle AB \rangle_{r-s+1}
\]

where \( \langle M \rangle_t \) is used to denote the t-grade part of multivector \( M \), e.g. consider the geometric product of two vectors \( ab = \langle ab \rangle_0 + \langle ab \rangle_2 = a \cdot b + a \wedge b \).

III. CONFORMAL GEOMETRY

Geometric algebra \( G_{4,1} \) can be used to treat conformal geometry in a very elegant way. To see how this is possible, we follow the same formulation presented in [2] and show how the Euclidean vector space \( \mathbb{R}^3 \) is represented in \( \mathbb{R}^{4,1} \). This space has an orthonormal vector basis given by \( \{e_i\} \) and \( e_{ij} = e_i \wedge e_j \) are bivectoral basis and \( e_{23}, e_{31} \) and \( e_{12} \)
correspond to the Hamilton basis. The unit Euclidean pseudo-scalar \( I_e := \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \), a pseudo-scalar \( I_c := I_e \mathbf{e}_5 \) and the bivector \( E := \mathbf{e}_4 \wedge \mathbf{e}_5 = \mathbf{e}_4 \mathbf{e}_5 \) are used for computing the inverse and duals of multivectors.

A. The Stereographic Projection

The conformal geometry is related to a stereographic projection in Euclidean space. A stereographic projection is a mapping taking points lying on a hypersphere to points lying on a hyperplane. In this case, the projection plane passes through the equator and the sphere is centered at the origin. To make a projection, a line is drawn from the north pole to each point on the sphere and the intersection of this line with the projection plane constitutes the stereographic projection.

For simplicity, we will illustrate the equivalence between stereographic projections and conformal geometric algebra in \( \mathbb{R}^1 \). We will be working in \( \mathbb{R}^{2,1} \) with the basis vectors \( \{e_1, e_4, e_5\} \) having the usual properties. The projection plane will be the x-axis and the sphere will be a circle centered at the origin with unitary radius.

![Stereographic projection for 1-D.](image)

Given a scalar \( x_c \) representing a point on the x-axis, we wish to find the point \( x_e \) lying on the circle that projects to it (see Figure 1). The equation of the line passing through the north pole and \( x_e \) is given by \( f(x) = \frac{-x - 1}{x} \) and the equation of the circle \( x^2 + f(x)^2 = 1 \). Substituting the equation of the line on the circle, we get the point of intersection \( x_c \)

\[
x_c = \left( \frac{2}{x_e^2 + 1}, \frac{x_e^2 - 1}{x_e^2 + 1} \right),
\]
which can be represented in homogeneous coordinates as the vector

\[
x_c = 2 \frac{x_e}{x_e^2 + 1} e_1 + \frac{x_e^2 - 1}{x_e^2 + 1} e_4 + e_5.
\]  

From (6) we can infer the coordinates on the circle for the point at infinity as

\[
e_\infty = \lim_{x_e \to -\infty} \{x_e\} = e_4 + e_5,
\]

\[
e = \frac{1}{2} \lim_{x_e \to 0} \{x_e\} = \frac{1}{2} (e_4 + e_5),
\]  

Note that (6) can be rewritten to

\[
x_c = x_e + \frac{1}{2} x_e^2 e_\infty + e_\infty.
\]  

B. Spheres and Planes

The equation of a sphere of radius \( p \) centered at point \( p_c \in \mathbb{R}^n \) can be written as

\[
(x_e - p_e)^2 = \rho^2.
\]  

Since \( x_c \cdot \mathbf{e}_4 = -\frac{1}{2} (x_e - y_e)^2 \), we can rewrite the formula above in terms of homogeneous coordinates as

\[
x_c \cdot p_c = -\frac{1}{2} \rho^2.
\]  

Since \( x_c \cdot \mathbf{e}_5 = -1 \) we can factor the expression above to

\[
x_c \cdot (p_c - \frac{1}{2} \rho^2 e_\infty) = 0.
\]  

Which finally yields the simplified equation for the sphere as \( s = p_c - \frac{1}{2} \rho^2 e_\infty \). Note from this equation that a point is just a sphere with zero radius. Alternatively, the dual of the sphere is represented as \( s^* = s I_c \). The advantage of the dual form is that the sphere can be directly computed from four points (in 3D) as

\[
s^* = x_c^1 A x_c^2 A x_c^3 A x_c^4.
\]  

If we replace one of these points for the point at infinity we get the equation of a plane

\[
\pi^* = x_c^1 A x_c^2 A x_c^3 A \mathbf{e}_5
\]

So that \( \pi \) becomes in the standard form

\[
\pi = I_c \pi^* = n + d e,
\]  

Where \( n \) is the normal vector and \( d \) represents the Hesse distance.

C. Circles and Lines

A circle \( z \) can be regarded as the intersection of two spheres \( s_1 \) and \( s_2 \) as \( z = (s_1 A s_2) \). The dual form of the circle (in 3D) can be expressed by three points lying on it as

\[
z^* = x_c^1 A x_c^2 A x_c^3.
\]  

Similar to the case of planes, lines can be defined by circles passing through the point at infinity as:

\[
L^* = x_c^1 A x_c^2 A \mathbf{e}_4
\]  

The standard form of the line (in 3D) can be expressed by

\[
L = l + \mathbf{e}_\infty (t \cdot l),
\]  

the line in the standard form is a bivector, and it has six parameters (Plucker coordinates), but just four degrees of freedom.

IV. DIRECT KINEMATICS

The direct kinematics involves the computation of the position and orientation of the end-effector given the parameters of the joints. The direct kinematics can be easily computed given the lines of the axes of screws.
A. Rigid Transformations

We can express rigid transformations in conformal geometry carrying out reflections between planes.

1) Reflection: The reflection of conformal geometric entities help us to do any other transformation. The reflection of a point \( x \) respect to the plane \( \pi \) is equal \( x \) minus twice the direct distance between the point and plane see the image (2), that is \( x' = x - 2(\pi \cdot x)\pi^{-1} \) to simplify this expression recalling the property of Clifford product of vectors \( 2(b \cdot a) = ab + ba \).

The reflection could be written

\[
x' = x - (\pi x - x \pi)\pi^{-1}, \quad (19)
\]

\[
x' = x - \pi x \pi^{-1} - 2x \pi \pi^{-1}, \quad (20)
\]

\[
x' = -\pi x \pi^{-1}. \quad (21)
\]

For any geometric entity \( Q \), the reflection respect to the plane \( \pi \) is given by

\[
Q' = \pi Q \pi^{-1} \quad (22)
\]

2) Translation: The translation of conformal entities can be by carrying out two reflections in parallel planes \( \pi_1 \) and \( \pi_2 \) see the image (3), that is

\[
Q' = \frac{(\pi_2 \pi_1)Q(\pi_1^{-1} \pi_2^{-1})}{T_\alpha} \quad (23)
\]

\[
T_\alpha = (n + d e_\infty)n = 1 + \frac{1}{2}a e_\infty = e^{-\frac{\theta}{2}} e_\infty \quad (24)
\]

With \( a = 2dn \).

3) Rotation: The rotation is the product of two reflections between nonparallel planes see image (4)

\[
Q' = \frac{(\pi_2 \pi_1)Q(\pi_1^{-1} \pi_2^{-1})}{R_\theta} \quad (25)
\]

Or computing the conformal product of the normals of the planes.

\[
R_\theta = n_2 n_1 = \cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})l = e^{-\frac{\theta}{2}l} \quad (26)
\]

With \( l = n_2 \wedge n_1 \), and \( \theta \) twice the angle between the planes \( \pi_2 \) and \( \pi_1 \). The screw motion called motor related to an arbitrary axis \( L \) is \( M = TRT \)

\[
Q' = \frac{(TRT_\theta)Q((TRT_\theta)^{-1})}{M_\theta} \quad (27)
\]

\[
M_\theta = TRT = \cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})L = e^{-\frac{\theta}{2}L} \quad (28)
\]

B. Kinematic Chains

The direct kinematics for serial robot arms is a succession of motors and it is valid for points, lines, planes, circles and spheres.

\[
Q' = \prod_{i=1}^{n} M_i Q \prod_{i=1}^{n} \tilde{M}_{n-i+1} \quad (29)
\]

V. DIFFERENTIAL KINEMATICS

The direct kinematics equation (29) can be used for points as.

\[
x'_p = \prod_{i=1}^{n} M_i x_p \prod_{i=1}^{n} \tilde{M}_{n-i+1} \quad (30)
\]

This equation could be used in conformal geometric algebra, using motors to represent 3D rigid transformations similar as with the motor algebra [4]. Now we produce an expression.
for differential kinematics through the total differentiation of (30) as follows.

$$dx' = \sum_{j=1}^{n} \partial_{q_j} \left( \prod_{i=1}^{n} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) dq_j \quad (31)$$

Each term of the sum is the product of two functions in $q_j$ then the differential reads,

$$dx' = \sum_{j=1}^{n} \left[ \partial_{q_j} \left( \prod_{i=1}^{j} M_i \prod_{i=j+1}^{n} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) + \prod_{i=1}^{n} M_i \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \partial_{q_j} \left( \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right) \right] dq_j. \quad (32)$$

Since $M = e^{-\frac{1}{2}q_L}$, the differential of the motor is $d(M) = -\frac{1}{2} M L dq$, thus we can write the partial differential of the motor’s product as follows.

$$\partial_{q_j} \left( \prod_{i=1}^{n} M_i \right) = -\frac{1}{2} \prod_{i=1}^{n-j} M_i L_j = -\frac{1}{2} \left( \prod_{i=1}^{j-1} M_i \right) L_j M_j \quad (33)$$

Similarly the differential of the $\tilde{M} = e^{\frac{1}{2}q_L}$ give us $d(\tilde{M}) = \frac{1}{2} M L dq$ and the differential of the product is.

$$\partial_{q_j} \left( \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right) = \frac{1}{2} \tilde{M}_j L_j \prod_{i=n-j+2}^{n} \tilde{M}_{n-i+1} \quad (34)$$

Replacing (33) and (34) in (32) we get,

$$dx' = \sum_{j=1}^{n} \left[ -\frac{1}{2} \prod_{i=1}^{j-1} M_i L_j M_j + \frac{1}{2} \prod_{i=1}^{n} M_i \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right] dq_j \quad (35)$$

which can be further simplified as

$$dx' = -\frac{1}{2} \sum_{j=1}^{n} \left[ \left( \prod_{i=1}^{j-1} M_i \right) L_j \left( \prod_{i=j+1}^{n} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) \right] dq_j \quad (36)$$

Note that the product of a vector with an $r$-vector is given by

$$a \cdot B_r = \frac{1}{2} \left( ab_r + (-1)^{r+1} b_r a \right). \quad (37)$$

Using the equation (37) we can simplify (36), since $L$ is a bivector and $x_p$ is a vector then we rewrite (36) as follows:

$$dx' = \sum_{j=1}^{n} \left[ \left( \prod_{i=1}^{j-1} M_i \right) \left( \prod_{i=j+1}^{n} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) \cdot L_j \left( \prod_{i=j+1}^{n} M_j \right) \right] dq_j \quad (38)$$

Similar as the case of points all the transformations in conformal geometric algebra can be also applied to the lines, thus

$$dx' = \sum_{j=1}^{n} \left[ \left( \prod_{i=1}^{j-1} M_i \prod_{i=j+1}^{n} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) \cdot \left( \prod_{i=j+1}^{n} M_j L_j \right) \left( \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right) \right] dq_j \quad (39)$$

Since $\prod_{i=1}^{n} M_i \prod_{i=j+1}^{n} M_i = \prod_{i=1}^{n} M_i$ we have

$$dx' = \sum_{j=1}^{n} \left[ \left( \prod_{i=1}^{j-1} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) \cdot \left( \prod_{i=j+1}^{n} M_j \right) \left( \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right) \right] dq_j \quad (40)$$

Recall the equation (30) of the direct kinematics, since in (40) appears again $x'_p$, we can replace (30) in (40) to get

$$dx' = \sum_{j=1}^{n} \left[ x'_p \cdot \left( \prod_{i=1}^{j-1} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \right) \cdot \left( \prod_{i=j+1}^{n} M_j \right) \left( \prod_{i=n-j+1}^{n} \tilde{M}_{n-i+1} \right) \right] dq_j \quad (41)$$

If we define $L' = \frac{1}{2} M L$ as function of $L$ as follows

$$L'_j = \prod_{i=1}^{j-1} M_i x_p \prod_{i=n-i+1}^{n} \tilde{M}_{n-i+1} \quad (42)$$

we get a very compact expression of differential kinematics.

$$dx' = \sum_{j=1}^{n} \left[ x'_p \cdot L'_j \right] dq_j \quad (43)$$

in this way we can finally write:

$$\dot{x}' = \left( \begin{array}{c} x'_p \cdot L'_1 \\ \vdots \\ x'_p \cdot L'_n \end{array} \right) \left( \begin{array}{c} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{array} \right) \quad (44)$$

**VI. KINEMATIC CONTROL FOR A PAN- TILT UNIT.**

We will show an example using our new formulation of the Jacobian. This is the control of a pan-tilt unit.

**A. The Pan-Tilt unit**

We implement algorithm for the velocity control of a pan-tilt unit (PTU Fig. 5) assuming three degree of freedom. We consider the stereo depth as one virtual D.O.F. thus the PTU has a similar kinematic behavior as a robot with three D.O.F.

In order to carry out a velocity control, we need first to compute the direct kinematics, this is very easy to do, as we know the axis lines:

$$L_1 = -e_{31} \quad (45)$$

$$L_2 = e_{12} + d_1 e_1 e_\infty \quad (46)$$

$$L_3 = e_1 e_\infty \quad (47)$$
Since $M_i = e^{-\frac{1}{2}q_i L_i}$ and $M_e = e^{\frac{1}{2}q_e L_i}$, we can compute the position of end effector using (30) as:

$$x_p(q) = x'_p = M_1M_2M_3x_p\tilde{M}_3\tilde{M}_2\tilde{M}_1,$$

(48)

The estate variable representation of the system is as follows

$$\begin{align*}
x'_p &= x' \cdot (L'_1 \quad L'_2 \quad L'_3) \\
y &= x'_p
\end{align*}$$  

(49)

where the position of end effector at home position $x_p$ is the conformal mapping of $x_{p} = d_{3}e_{1} + (d_{1} + d_{2})e_{2}$ (see eq. 9), the line $L'_i$ is the current position of $L_i$ and $u_i$ is the velocity of the $i$-junction of the system. As $L_3$ is an axis at infinity $M_3$ is a translator, that is, the virtual component is a prismatic junction.

**B. Exact linearization via feedback**

Now the following state feedback control law is chosen in order to get a new linear and controllable system.

$$\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = (x'_p \cdot L'_1 \quad x'_p \cdot L'_2 \quad x'_p \cdot L'_3)^{-1}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}$$  

(50)

Where $V = (v_1, v_2, v_3)^T$ is the new input to the linear system, then we rewrite the equations of the system

$$\begin{align*}
x'_p &= V \\
y &= x'_p
\end{align*}$$  

(51)

**C. Asymptotical output tracking**

The problem of follow a constant reference $x_t$ is solved computing the error between end effector position $x'_p$ and the target position $x_t$ as $e = (x'_p \wedge x_t) \cdot e$, the control law is then given by.

$$V = -ke$$  

(52)

This error is small if the control system is doing it's job, it is mapped to an error in the joint space using the inverse Jacobian.

$$U = J^{-1}V$$

Doing the Jacobian $J = x'_p \cdot (L'_1 \quad L'_2 \quad L'_3)$

$$j_1 = x'_p \cdot (L_1)$$

(54)

$$j_2 = x'_p \cdot (M_1 L_2 \tilde{M}_1)$$

(55)

$$j_3 = x'_p \cdot (M_1 M_2 L_3 \tilde{M}_2 \tilde{M}_1)$$

(56)

Once that we have the Jacobian is easy to compute the $dq_i$ using de crammers rule.

$$\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = (j_1 \wedge j_2 \wedge j_3)^{-1} \cdot \begin{pmatrix} 
V \wedge j_2 \wedge j_3 \\
(j_1 \wedge V) \wedge j_2 \wedge j_3 \\
(j_1 \wedge j_2) \wedge V
\end{pmatrix}$$  

(57)

This is possible because $j_1 \wedge j_2 \wedge j_3 = det(\bar{J})I_e$. Finally we have $dq_i$ which will tend to reduce these errors. Due to the fact that the Jacobian has singularities then we should use the pseudo inverse of Jacobian.

**D. Pseudo-inverse of Jacobian**

To avoid singularities we compute the pseudo inverse of Jacobian matrix

$$J = \begin{bmatrix} j_1 & j_2 \end{bmatrix}$$

(58)

Using the pseudo-inverse of Moore-Penrose

$$J^+ = (J^TJ)^{-1}J^T$$

(59)

Now evaluating $J$ in 59

$$J^+ = \frac{1}{det(J^TJ)} \begin{pmatrix} (j_2 \cdot j_3)j_1 - (j_2 \cdot j_1)j_3 \\
(j_1 \cdot j_1)j_2 - (j_2 \cdot j_1)j_1
\end{pmatrix}$$  

(60)

And Using Clifford algebra we could simplify further this equation

$$det(J^TJ) = (j_1 \cdot j_1)(j_2 \cdot j_2) - (j_1 \cdot j_2)^2$$  

(61)

$$= (j_1 \|j_2\|^2 - (j_1 \|j_2\|)^2 \cos^2(\theta)),$$  

(62)

$$= (j_1 \|j_2\|^2)^2 \sin^2(\theta),$$  

(63)

$$= |j_1 \wedge j_2|^2$$  

(64)

calling $\theta$ the angle between vectors. By the way each row of $J^+$ could be simplify as follows

$$\begin{pmatrix} (j_2 \cdot j_2)j_1 - (j_2 \cdot j_1)j_3 \\
(j_1 \cdot j_1)j_2 - (j_2 \cdot j_1)j_1
\end{pmatrix}$$  

(65)

$$\begin{pmatrix} (j_1 \cdot j_1)j_2 - (j_2 \cdot j_1)j_1 \\
(j_1 \cdot j_1)j_2 - (j_2 \cdot j_1)j_1
\end{pmatrix}$$  

(66)

Now the equation (59) can be rewritten as

$$J^+ = \frac{1}{|j_1 \wedge j_2|^2} \begin{pmatrix} (j_2 \cdot j_2)j_1 - (j_2 \cdot j_1)j_3 \\
(j_1 \cdot j_1)j_2 - (j_2 \cdot j_1)j_1
\end{pmatrix}$$  

(67)

Using this equation we can compute the input as $U = J^+V$ that is equal to

$$U = (j_1 \wedge j_2)^{-1} \cdot \begin{pmatrix} V \wedge j_2 \\
(j_1 \wedge V)
\end{pmatrix}$$  

(68)
E. Visual tracking

The target point is measured using two calibrated cameras (see Figure 6). With each image, we estimate the center of mass of the object in movement in order to do a retroprojection and finally estimate the 3D point. To compute the mass center, first we subtract the current image \( I \), an image in memory \( I_b \), the image in memory is the average of the last \( N \) images, this help us to substract the background.

\[
I_k(t) = I_a(t) - I_b(t-1) \ast N \tag{69}
\]

\[
I_b(t) = (I_a(t-1) \ast N + I_b)/(N+1) \tag{70}
\]

After that the moments of \( x \) and \( y \) are computed they are divided by the mass (pixels in movement) which corresponds to the intensity difference between the current and the memory images. In this way the mass center is obtained.

\[
x_o = \frac{\int_0^n \int_0^m I_k y dx dy}{\int_0^n \int_0^m I_k x dx dy} \tag{71}
\]

\[
y_o = \frac{\int_0^n \int_0^m I_k x dx dy}{\int_0^n \int_0^m I_k x dx dy} \tag{72}
\]

When the camera moves the background changes and its necessary to reset the \( N \) to 0 to restart the process of track.

Once we found the points \((X_o, X'_o)\) in the images, we calculated the lines of retro-projection.

\[
L = X_o \wedge C_c \wedge e_\infty \tag{73}
\]

\[
L' = X'_o \wedge C_c \wedge e_\infty \tag{74}
\]

The point in 3D is the intersection of these lines, in case they are not intersect is in the center of the direct distance.

E. Experimental results

In this experiment the binocular head should smoothly track a target. The figures (7a, 8a, 9a) show the 3D coordinates of the focus of attention. The figures (7b, 8b, 9b) show examples of the image sequence. we can see that the curves of the 3D object trajectory are very rough, however the control rule manages to keep the trajectory of the pan-tilt unit smooth.

VI. CONCLUSION

We have shown that geometric algebra is a powerful mathematical framework which allows us to deal not only with the mechanical problems, but also with the computer vision ones, thus for this purpose we can use it as a unifying language. On the other hand, it allows us to treat isometric and revolute joints in the same way, which helps to simply the formulation of rotations with respect to arbitrary axis.

The reader can appreciate the usefulness of using this kind of algebra for the case of the velocity control problem for the object visual tracking using a pan-tilt unit.

REFERENCES