Abstract—We present an approach to ruled surfaces and complex 3D-curves using conformal geometric algebra. As robots and mechanisms move, any line attached to them will trace out a ruled surface. In robotics, the arm of a robot usually needs to follow a curve over a ruled surface or over the intersection of some of them. In this paper we present the way to model and to resolve these problems using algebra of incidence and rigid-motion transformations in the context of the conformal geometric algebra.

I. INTRODUCTION

The main contribution of this paper is the generation of ruled surfaces or some other non trivial three dimensional curves using the mathematical system of Conformal Geometric Algebra (CGA). CGA is a fusion of the mathematical system of the XIX century, the Clifford Geometric Algebra (GA), and the non-Euclidean Hyperbolic Geometry. Historically, GA and CGA has not been taken into consideration seriously by the scientific community, but now and after the work of David Hestenes [1] and Pertti Lounesto [2] it has been taking a new scope of perspectives, not only theoretically, but for new and innovative applications to physics, computer vision, robotics and neural computing. One of the critics against CGA is the wrong idea that this system can manipulate only basic entities (points, lines, planes and spheres) and therefore it won’t be useful to model general two and three dimensional objects, curves, surfaces or any other nonlinear entity required to solve a problem of a perception action system in robotics and computer vision.

However, in this paper we present the CGA in conjunction with the algebra of incidence [4] and rigid-motion transformations, to obtain several practical techniques in the resolution of problems of perception action systems for visual guided robotics. For example 3D motion guidance, reaching or 3D object manipulation.

There are several interest points to study this subject: as robots and mechanisms are moving, any line attached to them will be tracing out a ruled surface or some other high nonlinear 3D-curve; the industry needs to guide the arm of robots with a laser welding to joint two ruled surfaces; reaching and manipulating 3D-objects is one of the main task in robotics, and it is usual that these objects have ruled surfaces or revolution surfaces; to guide a robot’s arm over a critical 2D or 3D-curve or any other configuration constraint, and so forth. See Figure 1.

Fig. 1. A laser welding following a 3D-curve: the projection of a cycloidal curve over a sphere.

The organization of the paper is as follows: section two is a brief introduction in geometric algebra. Section three explains the CGA. Section four is about direct and inverse-motion transformations (reflection, translation and rotation) in the context of CGA. In section five we present the way that several ruled surfaces or complex three dimensional curves can be generated in a very simple way using CGA. In section six we present a simulation where the arm of a robot needs to follow a 3D-curve on a non-planar ruled surface. Finally, in section seven we present the conclusions.

II. GEOMETRIC ALGEBRA

The algebras of Clifford and Grassmann are well known to pure mathematicians, but since the beginning were abandoned by physicists in favor of the vector algebra of Gibbs, the commonly algebra used today in most areas of physics. The approach to Clifford algebra that we adopt here has been developed since the 1960’s by David Hestenes [1], [3], [4].

1) Basic definitions: Let $V^n$ be a vector space of dimension $n$. We are going to define and generate an algebra $G_n$, called geometric algebra. Let \{\(e_1, e_2, \ldots, e_n\)\} be a set of basis vectors of $V^n$. The scalar multiplication and sum in $G_n$ are defined in the usual way of a vector space. The product or geometric product of elements of the basis of $G_n$ will be simply denoted by juxtaposition. In this way, from any two basis vectors $e_j$ and $e_k$, a new element of the algebra is
obtained and denoted as \( e_j e_k \equiv e_{jk} \). The product of basis vectors is anticommutative,

\[
e_j e_k = -e_k e_j, \quad \forall j \neq k.
\] (1)

The basis vectors must square in \(+1\), \(-1\) or \(0\), this means that there are no-negative integers \( p \), \( q \) and \( r \) such that \( n = p + q + r \) and

\[
e_i e_i = e_i^2 = \begin{cases} +1 & \text{for } i = 1, \ldots, p \\ -1 & \text{for } i = p + 1, \ldots, p + q \\ 0 & \text{for } i = p + q + 1, \ldots, n \end{cases}
\] (2)

This product will be called the geometric product of \( G_1 \). With these operations \( G_n \) is an associative linear algebra with identity and it is called the geometric algebra or Clifford algebra of dimension \( n = p + q + r \), generated by the vector space \( V^n \). It is usual to write \( G_n \equiv G_{p,q,r} \).

The elements of this geometric algebra are called multivectors, because they are entities generated by the sum of elements of mixed grade of the basis set of \( G_n \), such as

\[
A = \langle A \rangle_0 + \langle A \rangle_1 + \ldots + \langle A \rangle_n,
\] (3)

where the multivector \( A \in G_n \) is expressed by the addition of its 0-vector part (scalar) \( \langle A \rangle_0 \), its 1-vector part (vector) \( \langle A \rangle_1 \), its 2-vector part (bivector) \( \langle A \rangle_2 \), its 3-vector part (trivector) \( \langle A \rangle_3 \), and its \( n \)-vector part \( \langle A \rangle_n \). In particular, for \( n = 3 \), the standard basis of the geometric algebra \( G_3 \) is the following set, of \( 2^3 = 8 \) elements,

\[
\{1, e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\}.
\] (4)

We call an \( r \)-blade or a blade of grade \( r \) to the geometric product of \( r \) linearly independent vectors.

It will be convenient to define other products between the elements of this algebra which will allow us to set up several geometric relations (unions, intersections, projections, etc.) between different geometric entities (points, lines, planes, spheres, etc.) in a very simple way.

Firstly, we define the inner product, \( \mathbf{a} \cdot \mathbf{b} \), and the exterior or wedge product, \( \mathbf{a} \wedge \mathbf{b} \), of any two 1-vectors \( \mathbf{a}, \mathbf{b} \in G_3 \), as the symmetric and antisymmetric parts of the geometric product \( \mathbf{a} \mathbf{b} \), respectively. That is, from the expression

\[
\mathbf{a} \mathbf{b} = \frac{1}{2} (\mathbf{a} \mathbf{b} + \mathbf{b} \mathbf{a}) + \frac{1}{2} (\mathbf{a} \mathbf{b} - \mathbf{b} \mathbf{a})
\] (5)

we define the inner product

\[
\mathbf{a} \cdot \mathbf{b} \equiv \frac{1}{2} (\mathbf{a} \mathbf{b} + \mathbf{b} \mathbf{a})
\] (6)

and the outer or wedge product

\[
\mathbf{a} \wedge \mathbf{b} \equiv \frac{1}{2} (\mathbf{a} \mathbf{b} - \mathbf{b} \mathbf{a}).
\] (7)

Thus, we can express now the geometric product of two vectors in terms of these two new operations as

\[
\mathbf{a} \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.
\] (8)

From (6) and (7), \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \) and \( \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \).

Now, we can define these two new operations for the more general entities of \( G_n \), multivectors. For any two homogeneous multivectors \( \mathbf{A}_r \) and \( \mathbf{B}_s \) of grades \( r \) and \( s \), respectively, we define the inner product

\[
\mathbf{A}_r \cdot \mathbf{B}_s \equiv \begin{cases} \langle \mathbf{A}_r \mathbf{B}_s \rangle_{|r-s|} & \text{if } r > 0 \text{ and } s > 0 \\ 0 & \text{if } r = 0 \text{ or } s = 0 \end{cases}
\] (9)

and the outer or wedge product

\[
\mathbf{A}_r \wedge \mathbf{B}_s \equiv \langle \mathbf{A}_r \mathbf{B}_s \rangle_{r+s}.
\] (10)

A multivector \( \mathbf{A} \in G_n \) is called homogeneous of grade \( r \) if \( \mathbf{A} = \langle \mathbf{A} \rangle_r \).

The dual of a multivector \( \mathbf{A} \in G_n \) is defined by

\[
\mathbf{A}^* = \mathbf{A} \mathbf{I}_n^{-1}
\] (11)

where \( \mathbf{I}_n \equiv e_{12} \ldots n \) is the unit pseudoscalar of \( G_n \), and the inverse of a multivector \( \mathbf{A}_n \), if it exists, is defined by the equation \( \mathbf{A}^{-1} \mathbf{A} = 1 \).

III. CONFORMAL GEOMETRIC ALGEBRA

The geometric algebra of a 3D Euclidean space \( G_{3,0,0} \) has a point basis and the motor algebra \( G_{3,0,1} \), a line basis. In the latter the lines expressed in terms of Plücker coordinates can be used to represent points and planes as well, [7], [8]. In CGA the unit element is the sphere, which will allow us to represent other entities. We begin giving an introduction in conformal geometric algebra following the same formulation presented in [3], [6] and show how the Euclidean vector space \( \mathbb{R}^n \) is represented in \( \mathbb{R}^{n+1,1} \).

Let \( \mathbb{R}^{n+1,1} \) be the vector space with an orthonormal vector basis given by \( \{e_1, \ldots, e_n, e_+, e_-\} \), with the properties

\[
e_i^2 = 1, \quad e_+^2 = 1, \quad e_-^2 = -1,
\] (12)

\[
e_i \cdot e_+ = e_-, \quad e_+ \cdot e_- = 0
\] (13)

for \( i = 1, \ldots, n \). Note that this basis is not written in bold.

A null basis \( \{e_0, e_\infty\} \) can be introduced by

\[
e_0 = \frac{1}{2} (e_- - e_+),
\] (14)

\[
e_\infty = e_- + e_+
\] (15)

with the properties

\[
e_0^2 = e_\infty^2 = 0, \quad e_0 \cdot e_\infty = -1.
\] (16)

A unit pseudoscalar \( \mathbf{E} \in \mathbb{R}^{1,1} \) which represents the Minkowski plane is defined by \( \mathbf{E} \equiv e_\infty \wedge e_0 = e_+ \wedge e_- = e_+ e_- \). From here \( \mathbf{E}^2 = 1 \).

One of the results of the non-euclidean geometry demonstrated by Nikolai Lobachevsky in the XIX century is that in spaces with hyperbolic structure we can find subsets which are isomorphic to an euclidean space. In order to do this, Lobachevsky introduced two constraints, to the now called conformal point \( x_c \in \mathbb{R}^{n+1,1} \). See Figure 2. The first constraint is the homogeneous representation, normalizing the vector \( x_c \) such that

\[
x_c \cdot e_\infty = -1,
\] (17)
and the second constraint is such that the vector must be a null vector, that is,
\[ x_c^2 = 0. \]  
(18)

Fig. 2. The Null Cone and the Horosphere for 1-D, and the conformal and stereographic representation of a 1-D vector.

Thus, conformal points are required to lie in the intersection surface, denoted \( \mathbb{N}^n_c \), between the null cone \( \mathbb{N}^{n+1} \) and the hyperplane \( \mathbb{P}(e_\infty, c_0) \):
\[
\mathbb{N}^n_c = \mathbb{N}^{n+1} \cap \mathbb{P}(e_\infty, c_0) = \{ x_c \in \mathbb{R}^{n+1} | x_c^2 = 0, x_c \cdot e_\infty = -1 \}. \]  
(19)

The constraint (19) define an isomorphic mapping between the Euclidean and Conformal spaces. Thus, for each conformal point \( x_c \in \mathbb{R}^{n+1} \) there is a unique euclidean point \( x_e \in \mathbb{R}^n \) and unique scalars \( \alpha, \beta \) such that the mapping \( x_c \mapsto x_e + \alpha e_0 + \beta e_\infty \) is bijective. From (17) and (18) we can obtain \( \alpha = 1 \) and \( \beta = \frac{1}{2} x_c^2 \). Then, the standard form of a conformal point \( x_c \) is
\[ x_c = x_e + \frac{1}{2} x_c^2 e_\infty + e_0. \]  
(20)

IV. DIRECT AND OPPOSITE-MOTION TRANSFORMATIONS

In the middle of the XIX century J. Liouville proved, for the 3-dimensional case, that any conformal mapping on the whole of \( \mathbb{R}^n \) can be expressed as a composite of inversions in spheres and reflections in hyperplanes. [9] In particular, rotation, translation, dilation and inversion mappings will be obtained with these two mappings. CGA simplifies this concepts because the isomorphism between the conformal group on \( \mathbb{R}^n \) and the Lorentz group on \( \mathbb{R}^{n+1} \) help us to express with a linear Lorentz transformation a non-linear conformal transformation, and then to use versor representation to simplify composition of transformations with multiplication of vectors. [3] Thus, using CGA is computationally more efficient and simpler to interpret the geometry of the conformal mappings, than with matrix algebra. A transformation of geometric figures is said to be conformal if it preserves the shape of the figures, that is, whether it preserves the angles and hence the shapes of straight lines and circles. In particular, rotation and translation mappings are conformal and are also called direct-motion transformations. Inversion and reflection mappings preserve the magnitude of the angle but reverse its direction and then they are also called opposite-motion transformations.

From [3] a conformal transformation in the geometric algebra framework uses a versor representation as
\[ g(x_c) = G x_c (G^*)^{-1} = \sigma x_c^*, \]  
(21)
where \( x_c \in \mathbb{R}^{n+1} \), \( G \) is a versor (a multivector that can be expressed as the geometric product of invertible vectors), and \( \sigma \) is a scalar. \( G \) can be expressed in geometric algebra as a composite of versors for inversions in spheres and reflections in hyperplanes.

1) Inversion with respect to a sphere: By the classical definition, an inversion \( T_S \) with respect to a sphere \( S \) (of radius \( \rho \) and centre \( c \)) is such that for any point \( q \) at a distance \( d \) from \( c \), \( T_S(q) \) will be in the same ray from \( c \) to \( q \) and at a distance \( \rho^2/d \) from \( c \). We comment some of the powerful properties of this transformation: (a) The inverse of a plane through the center of inversion is the plane itself. (b) The inverse of a plane not passing through the center of inversion is a sphere passing through the center of inversion. (c) The inverse of a sphere through the center of inversion is a plane not passing through the center of inversion. (d) The inverse of a sphere not passing through the center of inversion is a sphere not passing through the center of inversion. Figure 3. (e) Inversion in a sphere maps lines and circles to lines and circles.

Fig. 3. The inversion with respect to a sphere \( T \) mapping sphere to sphere and circle to circle, if they are not passing through the center of \( T \).

In the context of the conformal geometry, the general form of a reflection about a vector \( s \) is
\[ s(x_c) = -sx_c s^{-1} = x_c - 2(s \cdot x_c) s^{-1} = \sigma x_c^*, \]  
(22)
where \( sx + xs = 2(s \cdot x) \), using the definition of the Clifford product between two vectors. Now, in the conformal geometric framework the versor \( s \) in (21) can be any versor, in particular a sphere! Then using (21) we can obtain the inversion-reflexion of an entity with respect to a sphere and not only with respect to a single vector.

From [3] we know that in CGA the equation of a sphere of radius \( \rho \) centered at point \( c_e \) is the conformal vector
\[ s = c_e - \frac{1}{2} \rho^2 e_\infty. \]  
(23)

2) Reflections: A hyperplane with unit normal \( n \) and signed distance \( \delta \) from the origin in \( \mathbb{R}^n \) can be represented by the vector
\[ s = n + \delta e_\infty. \]  
(24)
Inserting $s \cdot x_e = n \cdot x_e$ into Eq. (22), we find that

$$g(x_e) = nx_e n^\perp + 2\delta n = n(x_e - \delta n)n^\perp + \delta n,$$

(25)

where $n^\perp = -n^{-1}$. This expression is equivalent to a reflection $nx_e n^\perp$ at the origin, translated by $\delta$ along the direction of $n$. A point $c_e$ is on the hyperplane when $\delta = n \cdot c_e$, in which case Eq. (24) can be written as

$$s = n + e_\infty n \cdot c_e.$$  

(26)

Via Eq. (25), this vector represents the reflection in a hyperplane through point $c_e$. 

Fig. 4. The translation from $X$ to $T_2(T_1(X))$ is obtained as the inversion-reflection with two parallel planes. $\Pi_1$ passes through the origin.

3) **Translations**: Translations can be modeled by two reflections, one reflection with respect to a plane passing through the origin and the other to a distance $\delta$ from the origin. Then, from Eq. (24), we can represent the operator for a translation (called a translator) as

$$T_a = \pi_1 \pi_2 = (n + \delta e_\infty)(n + 0 e_\infty),$$

$$= 1 + \frac{1}{2} a e_\infty,$$

(27)

where $a = 2\delta n$ and $\|n\| = 1$. The translation distance is twice the separation between the hyperplanes. See Figure 4.

Fig. 5. The rotation as the inversion-reflection with two non-parallel planes.

4) **Rotations**: Rotations can be modeled by the composition of two reflections about two hyperplanes intersecting in a common point $c_e$, as in

$$R = (a + e_\infty a \cdot c_e)(b + e_\infty b \cdot c) = ab + e_\infty c_e \cdot (a \wedge b),$$  

(28)

where $a$ and $b$ are unit normals. See Figure 5. Rotations about the origin can also be written in exponential form, as in

$$R = e^{\frac{1}{2}aB},$$

(29)

where $B$ is a unit bivector, such that the dual $B^*$ is the axis of rotation and $\alpha$ is the magnitude of the angle of rotation.

In general we can write down a canonical decomposition of a motor $M$ for a rigid motion in terms of two versors, translation and rotation, as

$$M = T_2 R_\alpha.$$  

(30)

V. **Ruled Surfaces**

Conics, ellipsoids, helicoids, hyperboloid of one sheet are entities which can not be directly described in CGA, however, can be modeled with its multivectors. In particular, a ruled surface is a surface generated by the displacement of a straight line (called generatrix) along a directing curve or curves (called directrices). The plane is the simplest ruled surface, however we can generate complex surfaces. For example, a circular cone is a surface generated by a straight line through a fixed point and a point in a circle. It is well known that the intersection of a plane with the cone can generate the conics. See Figure 6. In [10] the cycloidal curves can be generated by two coupled twists. In this section we are going to see how these and other curves and surfaces can be obtained using only multivectors of CGA.

Fig. 6. Hyperbola as the *meet* of a cone and a plane.

1) **Cone and Conics**: A circular cone is described by a fixed point $v_0$ (vertex), a dual circle $z = a_0 \wedge a_1 \wedge a_2$ (directrix) and a rotor $R(\theta, l), \theta \in [0, 2\pi)$ rotating the straight line $L(v_0, a_0) = v_0 \wedge a_0 \wedge e_\infty$, (generatrix) along the axis of the cone $l = z \cdot e_\infty$. Then, the cone $w$ is generated as

$$w = R(\theta, l_0) L(v_0, a_0) R(\theta, l_0), \theta \in [0, 2\pi)$$  

(31)

A conic curve can be obtained with the *meet* of a cone and a plane. See Figure 6.

2) **Cycloidal Curves**: The family of the cycloidal curves can be generated by the rotation and translation of one or two circles. For example, the cycloidal family of curves generated by two circles of radius $r_0$ and $r_1$ are expressed by, see Figure 1, the motor

$$M = TR_1 T^* R_2$$  

(32)
where
\[
T = T((r_0 + r_1)(\sin(\theta)e_1 + \cos(\theta)e_2)) \tag{33}
\]
\[
R_1 = R_1\left(\frac{r_0}{r_1}\right) \tag{34}
\]
\[
R_2 = R_2(\theta) \tag{35}
\]

Then, each conformal point \(x\) is transformed as \(M\tilde{M}\).

3) Helicoid: We can obtain the ruled surface called helicoid rotating a ray segment in a similar way as the spiral of Archimedes. So, if the axis \(e_3\) is the directrix of the rays and orthogonal to them, then the translator that we need to apply is a multiple of \(\theta\), the angle of rotation. See Figure 7.

![Fig. 7. The helicoid is generated by the rotation and translation of a line segment. In CGA the motor is the desired multivector.](image)

4) Sphere and Cone: Let us see an example of how the algebra of incidence using CGA simplify the algebra. The intersection of a cone and a sphere in general position, that is, the axis of the cone does not pass through the center of the sphere, is the three dimensional curve of all the euclidean points \((x, y, z)\) such that \(x\) and \(y\) satisfy the quartic equation

\[
[x^2(1 + \frac{1}{c^2}) - 2x_0x + y^2(1 + \frac{1}{c^2}) - 2y_0y + \\
+x_0^2 + y_0^2 + z_0^2 - r^2]^2 = 4z_0^2(x^2 + y^2)/c^2 \tag{36}
\]

and \(x, y\) and \(z\) the quadratic equation

\[
(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2. \tag{37}
\]

See Figure 8. In CGA the set of points \(q\) of the intersection can be expressed as the meet of the dual sphere \(s\) and the cone \(w\), (31), defined in terms of its generatrix \(L\), that is

\[
q = (s^*) \cdot \left[ R(\theta, l_0) L(v_0, a_0) \tilde{R}(\theta, l_0) \right], \quad \theta \in [0, 2\pi). \tag{38}
\]

Thus, in CGA we only need (38) to express the intersection of a sphere and a cone, meanwhile in euclidean geometry it is necessary to use (36) and (37).

5) Hyperboloid of one sheet: The rotation of a line over a circle can generate a hyperboloid of one sheet. Figure 9.

![Fig. 9. Hyperboloid as the rotor of a line.](image)

6) Ellipse and Ellipsoid: The ellipse is a curve of the family of the cycloid and with a translator and a dilator we can obtain an ellipsoid.

7) Plücker Conoid: The cylindroid or Plücker conoid is a ruled surface. See Figure 10. This ruled surface is like the helicoid where the translator parallel to the axis \(e_3\) is of magnitude, a multiple of \(\cos(\theta)\sin(\theta)\).

![Fig. 10. The Plücker conoid as a ruled surface.](image)

VI. RESULTS

We proceed to show simulated results of the ideas presented in this paper. Consider a robot arm laser welder. See Figure 11. The welding distance has to be kept constant and the end-effector should follow a 3D-curve \(w\) on the ruled surface guided only by the directrices \(d_1, d_2\) and a guide line \(L\). From the generatrices we can always generate the irregular ruled surface, and then with the meet with another surface we can obtain the desired 3D-curve. We tested our simulations with
several ruled surfaces obtaining expressions of very irregular surfaces and 3D curves that with the standard vector and matrix analysis it would be very difficult to obtain them.

Fig. 11. A laser welding following a 3D-curve $w$ on a ruled surface defined by the directrices $d_1$ and $d_2$. The 3D-curve $w$ is the meet between the ruled surface and a plane containing the line $L$.

VII. CONCLUSION

In this paper we present with several kinds of surfaces an approach to ruled surfaces and 3D curves using the algebra of incidence of CGA. The scientific community has the wrong idea that CGA can generate only basic entities. This paper gives us a first approximation to model and generate very complex surfaces and curves using multivectors and the three products of CGA, geometric, inner and wedge product, to generate the desired results.

REFERENCES


