Lyapunov–Krasovskii functionals for scalar neutral type time delay equations

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\textbf{A B S T R A C T}

In this paper a procedure for construction of complete type Lyapunov–Krasovskii functionals for a scalar neutral type time delay equation is considered. The construction of the functionals depends on the so-called Lyapunov functions which satisfy a delay equation with additional boundary conditions. It is shown that these functionals admit lower and upper quadratic bounds. Exponential estimates for solutions of the scalar neutral type time delay equations based on the Lyapunov–Krasovskii functionals are presented.

A new definition of the Lyapunov function is given, and a detailed analysis of its properties is carried out.

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\section{1. Introduction}

Neutral type time delay systems form an important class of time delay systems where the system dynamics depend both on the delayed state and its derivative. It is well known that the Lyapunov–Krasovskii functional approach plays an important role in the stability analysis of time delay systems, see survey papers \cite{6,10}. However, there are serious limitations in the practical application of this approach. One of the limitations is a lack of efficient algorithms for constructing the corresponding Lyapunov–Krasovskii functionals. Despite the fact that there are general expressions for the functionals, the common practice consists in exploiting various reduced type functionals, see for instance \cite{3}. In other words, functionals are not computed for a given neutral type time delay system, but rather the system is adjusted for the proposed functionals. This is justified by severe difficulties arising in computing of appropriate Lyapunov–Krasovskii functionals for a given neutral type time delay system. On the other hand, the practice implies that there is no guarantee that such reduced type functionals may serve for stability analysis of a given neutral type time delay system. The construction of Lyapunov–Krasovskii functionals with a prescribed time derivative for time-invariant linear delay systems was initiated in \cite{11}. Since then several interesting results have been reported in \cite{2,5,4}. Recently, these contributions were clarified and completed in \cite{9} by proposing a modified Lyapunov–Krasovskii functional with a prescribed time derivative.

The modified functionals were called as complete type ones. The complete type functionals admit quadratic lower bounds. A special matrix-valued function represents a fundamental element of the procedure for the construction of the complete type functionals. For a time delay system, the matrix-valued function plays exactly the same role as the solution of the classical Lyapunov matrix equation plays in computing of Lyapunov quadratic functions for a delay-free system. In \cite{7} for the case of neutral type time delay systems with one delay, quadratic Lyapunov functionals with a given time derivative have been presented. In this contribution, in the case of a scalar neutral type time delay equation with several delays, we study Lyapunov–Krasovskii functionals with a given time derivative, the construction of the functionals depends on a scalar function which satisfies a delay equation with additional boundary conditions, and we consider a useful application of the functionals, the exponential estimates for the solutions of the scalar neutral type time delay equations.

The paper is organized as follows. In Section 2 we introduce some preliminaries on scalar neutral type time delay equations. An explicit expression of a quadratic Lyapunov functional, which is the basis for the construction of the complete type functionals, and the corresponding Lyapunov functions are introduced here.

In Section 3, firstly, we present some properties for the Lyapunov function under the assumption that the neutral time delay equation is exponentially stable, see Theorem 1 and Lemma 2. Then, a new definition of the Lyapunov function which does not demand stability of the scalar neutral type time delay equations is presented. At last, it is shown that the Lyapunov function can be computed as a solution of a special two point boundary problem for a delay-free system of linear equations, see Theorem 3. In Section 4 a general expression for the complete type Lyapunov–Krasovskii

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functional is given, and some lower and upper bounds for the functionals are obtained. In Section 5 a constructive procedure for deriving exponential estimates for the solutions of the scalar neutral type time delay equations based on the functionals is proposed. In Section 6 a numerical example illustrates the principal results of the paper. The contribution ends with some concluding remarks.

2. Basic facts

In this note a scalar neutral type time delay equation of the form

\[ \sum_{j=0}^{m} a_j x(t - jh) = \sum_{j=0}^{m} b_j x(t - jh), \quad t \geq 0, \quad (1) \]

will be considered. Here \( a_j, b_j \in \mathbb{R}, j = 0, \ldots, m, \) are real coefficients \((a_0 = 1), \) and \( h > 0 \) is a given basic time delay.

For an initial function \( \varphi : [-mh, 0] \to \mathbb{R} \) there exists the unique solution, \( x(t), \varphi, \) of (1) such that \( x(\theta, \varphi) = \varphi(\theta), \theta \in [-mh, 0]. \) It is assumed here that \( \varphi \) is a continuously differentiable function defined on the interval \([-mh, 0], x(t) \) stands for the restriction of \( x(t, \varphi) \) to the interval \([t - mh, t], \) so that \( x_t \) is defined by \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-mh, 0]. \)

Definition 1 ([1]). Eq. (1) is said to be exponentially stable if there exist \( \gamma \geq 1 \) and \( \sigma > 0 \) such that every solution \( x(t), \varphi \) satisfies the inequality

\[ |x(t, \varphi)| \leq \gamma e^{-\sigma t} |\varphi|_{mh}, \quad t \geq 0, \]

where \( |\varphi|_{mh} = \max_{\varphi \in [-mh, 0]} |\varphi(\theta)|. \)

2.1. Fundamental solution

Definition 2 ([1]). Let \( k(t) \) be the solution of Eq. (1) that satisfies the conditions

- \( k(t) = 0, \) for \( t < 0, \) and \( k(0) = 1; \)
- \( \sum_{j=0}^{m} a_j k(t - jh) \) is continuous for \( t \geq 0; \)
- the function \( k(t) \) is known as the fundamental solution of Eq. (1).

It follows from Definition 2 that the fundamental solution has jump discontinuities. The following statement describes the jumps.

Lemma 1 ([1]). The second condition of Definition 2 implies that the fundamental solution \( k(t) \) has jump discontinuities at the points \( t_i = ih, i = 0, 1, 2, \ldots \)

\[ \Delta k(t)|_{t=ih} = \Delta i = \lim_{t \to ih^+} k(t) - \lim_{t \to ih^-} k(t), \]

and \( k(t_i) = \lim_{t \to t_i^+} k(t). \) The jump discontinuities satisfy the difference equation

\[ \sum_{j=0}^{m} a_j \Delta_{i-j} = 0, \quad i = 1, 2, \ldots, \]

with the following initial condition: \( \Delta_{-m+1} = \cdots = \Delta_{-1} = 0, \) and \( \Delta_0 = 1. \)

2.2. Cauchy formula

Given an initial condition \( \varphi(\theta), \theta \in [-mh, 0], \) the fundamental solution \( k(t) \) provides the following explicit expression for the corresponding solution \( x(t, \varphi) \) of Eq. (1) for \( t \geq 0 \)

\[ x(t, \varphi) = \left[ \sum_{j=0}^{m} a_j k(t - jh) \right] \varphi(0) + \sum_{j=1}^{m} \int_{-jh}^{0} k(t - jh - \theta) \left[ b_j \varphi(\theta) - a_j \varphi(\theta + h) \right] d\theta. \quad (2) \]

The latter expression is known as the Cauchy formula for Eq. (1).

2.3. Functional \( v_0(x) \)

Let Eq. (1) be exponentially stable. Then, there exists a quadratic functional \( v_0(x) \) such that

\[ \frac{d}{dt} v_0(x) = -x^2(t), \quad t \geq 0. \]

The functional can be written as

\[ v_0(\varphi) = \int_{0}^{\infty} x^2(t) \, dt. \]

Substituting \( x(t, \varphi) \) under the integral on the right-hand side of the last equality by the Cauchy formula (2) one arrives at the following expression for the functional

\[ v_0(x) = \left( \sum_{j=0}^{m} \sum_{i=0}^{m} a_j a_i u((j-i)h) \right) x^2(t) + 2x(t) \sum_{j=0}^{m} \sum_{i=1}^{m} a_j \int_{-ih}^{0} u((j-i)h - \theta) \left[ b_j x(t + \theta) - a_j x(t + \theta + h) \right] d\theta + \int_{-mh}^{0} \int_{-mh}^{0} \left[ 2 a_j x(t + \theta_1) - a_j x(t + \theta_1 + h) \right] d\theta_2 d\theta_1. \quad (3) \]

Here, the scalar function

\[ u(\tau) = \int_{0}^{\infty} k(t) (t + \tau) \, dt, \quad (4) \]

is known as the Lyapunov function for Eq. (1). In order to compute \( v_0(x) \) one has to know \( u(\tau) \) for \( \tau \in [-mh, mh]. \)

3. Lyapunov function

It follows from expression (3) that the Lyapunov function is the key component in the construction of Lyapunov quadratic functionals. Nevertheless, the above construction of the Lyapunov functional [3] would not be practical if it required the evaluation of integral equation (4) (and so, the knowledge of the fundamental solution \( k(t) \) on \( \mathbb{R}_+ \)). To deal with this difficulty, first we will present several basic properties of the Lyapunov function. Then, we will show that the construction of these functionals is based on a solution of a scalar neutral type delay equation which satisfies additional symmetry and algebraic properties. These results allow us to present a new definition of the Lyapunov function. The rest of the section is devoted to the first derivative of the Lyapunov function and on the computational issue.

3.1. Properties of the Lyapunov function

The Lyapunov function satisfies the following statement.

Theorem 1. Let Eq. (1) be exponentially stable. The Lyapunov function (4) satisfies the following properties

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Dynamic property:
\[
\sum_{j=0}^{m} a_i u'(\tau - jh) = \sum_{j=0}^{m} b_i u(\tau - jh), \quad \tau \geq 0.
\]  
(5)

Symmetry property:
\[
u(-\tau) = u(\tau), \quad \tau \geq 0.
\]  
(6)

Algebraic property:
\[
\sum_{j=0}^{m} \sum_{i=0}^{m} a_i b_i u((j-i)h) = -1/2.
\]  
(7)

Proof. The first two properties follow directly from (4). For the algebraic property, we start with the equality
\[
\frac{d}{dt} \left( \sum_{j=0}^{m} a_i k(t - jh) \right)^2 = \left( \sum_{j=0}^{m} a_i k(t - jh) \right) \left( \sum_{i=0}^{m} b_i k(t - ih) \right) = \left( \sum_{j=0}^{m} \sum_{i=0}^{m} a_i b_i k(t - jh) k(t - ih) \right).
\]

Now, integrating this equality from 0 to \( \infty \) we arrive at the desired algebraic property. \( \square \)

3.2. New definition of the Lyapunov function

The previous properties of the Lyapunov function give rise to an alternative definition of the function, which does not demand exponential stability of the neutral type time delay equation. To arrive at the new definition it is worth to mention that the only properties of \( u(\tau) \) needed in order to show that the time derivative of \( v_{0}(x_{0}) \) coincides with \( -x^2(t) \) are those which are listed in (5)-(7). The next theorem is the basic result which states that the assumption of Eq. (1) be exponentially stable is not required.

Theorem 2. Let function \( \tilde{u}(\tau), \tau \in [-mh, mh] \), satisfy the three properties (5)-(7). Then, the time derivative of the functional (3) with \( u(\tau) = \tilde{u}(\tau) \) is equal to \( -x^2(t) \).

Proof. The statement follows directly when we derive functional (3), employing the three properties (5)-(7). \( \square \)

This theorem gives rise to the following alternative definition of the Lyapunov function which does not demand the exponential stability of Eq. (1).

Definition 3. A Lyapunov function for Eq. (1) is a solution of Eq. (5) which satisfies the properties (6) and (7).

With this new definition the knowledge of the fundamental solution \( k(t) \) on \( \mathbb{R}_+ \) is not necessary.

3.3. First derivative of the Lyapunov function

As it has been mentioned the fundamental solution of Eq. (1) has jump discontinuities at points \( t_i = ih, i = 0, 1, 2, \ldots \), this discontinuity implies that the first derivative of the Lyapunov function suffers discontinuity, too.

Lemma 2. Let Eq. (1) be exponentially stable. Then the first derivative of the Lyapunov function (4) satisfies the following relations
\[
\sum_{j=0}^{m} a_i \Delta u'(-jh) = -1, \quad \text{and}
\]
\[
\sum_{j=0}^{m} a_i \Delta u'(i-jh) = 0, \quad i = 1, 2, \ldots.
\]
Here \( \Delta u'(\tau) = \lim_{\chi \to 0^+} u'(\chi) - \lim_{\chi \to 0^-} u'(-\chi) \).

Proof. From Definition 2, it is known that \( \sum_{j=0}^{m} a_i k(t - jh) \) is continuous for \( t \geq 0 \). Let us integrate from 0 to \( \infty \) the following expressions (here, the integrals are Riemann–Stieltjes integrals):
\[
\sum_{j=0}^{m} a_i k(t + lh - jh + 0) \int_0^t dt(k(t))
\]
and
\[
\sum_{j=0}^{m} a_i k(t + lh - jh - 0) \int_0^t dt(k(t))
\]
here \( k(t + 0) = \lim_{\chi \to 0^+} k(\chi), k(t - 0) = \lim_{\chi \to 0^-} k(\chi) \) and \( i = 0, 1, 2, \ldots \). In performing the integration one should remember that \( k(t) \) has jump discontinuities at points \( t_i = ih, i = 0, 1, 2, \ldots \), and that \( k(t_i) = k(t_i + 0) \). The first integral,
\[
\int_0^\infty \left( \sum_{j=0}^{m} a_i k(t + lh - jh + 0) \right) \int_0^t dt(k(t))
\]
where \( \Delta t \) is the size of the jump of the fundamental solution at point \( t_i \), see Lemma 1. On the other hand, this integral can be also computed as follows
\[
\int_0^\infty \left( \sum_{j=0}^{m} a_i k(t + lh - jh + 0) \right) \int_0^t dt(k(t)) = -k(0) \left( \sum_{j=0}^{m} a_i k(lh - jh + 0) \right) k(0) - \int_0^\infty k(t) \left( \sum_{j=0}^{m} a_i k(t + lh - jh + 0) \right).
\]
As function \( k(t) \) satisfies Eq. (1), the integral term on the right-hand side of the last equality is equal to \( -\int_0^\infty k(t) (\sum_{j=0}^{m} a_i b_i k(t + lh - jh + 0)) dt \). Now using (4) and (5), one may deduce that
\[
\sum_{j=0}^{m} \int_0^\infty \left( \sum_{j=0}^{m} a_i b_i k(t + lh - jh + 0) \right) dt(k(t))
\]
\[
+ \sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_i k(lh - jh + 0) \right)
\]
\[
= -k(0) \left( \sum_{j=0}^{m} a_i k(lh - jh + 0) \right) + \sum_{j=0}^{m} a_i u'(lh - jh + 0).
\]  
(8)

Following a similar procedure for the second integral one obtains
\[
\sum_{j=0}^{m} \int_0^\infty \left( \sum_{j=0}^{m} a_i k(t + lh - jh - 0) \right) dt(k(t))
\]
+ \sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_j k (i + l + j) \right) \\
= - \left( \sum_{j=0}^{m} a_j k (l + j) \right) k(0) - \sum_{j=0}^{m} a_j u(t) (l + j).
\tag{9}

Subtracting (9) from (8) the following equality occurs
\[
\sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_j k (i + l + j) \right) \mathrm{dk}(t) + \sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_j k (i + l + j - i) \right) = -k(0) \sum_{j=0}^{m} a_j k (l + j).
\]

It is worth to mention that \( \sum_{j=0}^{m} a_j k (i + l + j) = 0 \) for \( t > 0 \), so
\[
\sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_j k (i + l + j) \right) \mathrm{dk}(t) = 0.
\]

By Lemma 1 the sum \( \sum_{i=1}^{\infty} \Delta_i \left( \sum_{j=0}^{m} a_j k (i + l + j - i) \right) = 0 \), because here \( i + l > 0 \).

As a result one arrives at the equality
\[
\sum_{j=0}^{m} a_j k (l + j) h = -k(0) \sum_{j=0}^{m} a_j k (l + j).
\]

Now, employing once again Lemma 1, and the fact that \( k(0) = 1 \), the lemma statement follows.

It follows from Lemma 2 that \( u'(t) \) has jump discontinuities. The following statement provides the size of the jumps of the first derivative of the Lyapunov function.

**Lemma 3** \([7]\). The jump size of the first derivative of the Lyapunov function \( u'(t) \) can be computed as follows
\[
\left\{ \begin{array}{l}
\Delta u'(lh) = -e_1^T P s e_1, \quad l = 0, 1, 2, \ldots, \\
\Delta u'(-lh) = -e_1^T (S^T)^p e_1, \quad l = 1, 2, \ldots,
\end{array} \right.
\]
where \( e_1 = [1 \ 0 \ \cdots \ 0] \), matrix
\[
S = \begin{bmatrix}
-a_1 & -a_2 & \cdots & -a_{m-1} & -a_m \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\]
and matrix \( P \) is the solution of the matrix equation \( S^T P S - P = -W_0 \) with \( W_0 = e_1 e_1^T \).

### 3.4 Computational issue

As it has been shown in Theorem 2 one has to know the function \( u(t) \) which satisfies conditions (5)-(7) in order to construct the Lyapunov–Krasovskii functional (3). So, in this section we present a procedure for the computation of the function.

Let us introduce \( 2m \) auxiliary functions as follow
\[
z_k(t) = u(t + kh), \quad t \in [0, h], \quad k = -m, -m + 1, \ldots, -1, 0, 1, \ldots, m - 1,
\]
and define the vector
\[
\tilde{z}(t) = \begin{bmatrix} z_{m-1}(t) & \cdots & z_0(t) & \cdots & z_{-m}(t) \end{bmatrix}^T.
\]

We also need matrices
\[
A = \begin{bmatrix}
a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\
0 & a_0 & \cdots & a_{m-1} & a_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & a_m & \cdots & a_1 & a_0 & \cdots & 0 \\
0 & 0 & \cdots & a_{m-1} & a_{m-2} & \cdots & a_0
\end{bmatrix}
\]
and
\[
B = \begin{bmatrix}
b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\
0 & b_0 & \cdots & b_{m-1} & b_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & -b_m & \cdots & -b_0 & b_m & \cdots & 0 \\
0 & -b_m & \cdots & -b_1 & -b_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_{m-1} & -b_{m-2} & \cdots & -b_0
\end{bmatrix}.
\]

**Remark 1.** It is worth to be mentioned that matrix \( A \) is the resultant matrix of the polynomials \( p(s) \) and \( s^m p(s) \), where
\[
p(s) = a_0 s^m + a_1 s^{m-1} + \cdots + a_{m-1} s + a_m,
\]
while matrix \( B \) is the resultant matrix of the polynomials \( q(s) \) and \( -s^m q(s) \), where
\[
q(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m.
\]

**Theorem 3.** Let \( u(t) \) satisfy the three properties (5)-(7). Then, there exists a solution \( \tilde{z}(t) \) of the delay-free system
\[
A \frac{d}{dt} \tilde{z}(t) = B \tilde{z}(t),
\]
such that equalities (10) hold. The solution satisfies the following set of \( 2m \) boundary conditions
\[
\sum_{j=0}^{m} a_j z_{k+j}(0) = z_k(h), \quad k = -m, -m + 1, \ldots, -1, 0, 1, \ldots, m - 2;
\]
\[
\sum_{j=0}^{m} a_j z_{k-j}(h) = -1/2.
\]

**Proof.** The dynamic property (5) provides for \( t \in [0, h] \) the equalities
\[
\sum_{j=0}^{m} a_j z_{k-j}(t) = \sum_{j=0}^{m} b_j z_{k-j}(t), \quad k = 0, 1, \ldots, m - 1.
\]

For \( k < 0 \)
\[
z_k(t) = u(-\tau - k h), \quad \tau \to > 0
\]
and the dynamic property takes the form
\[
\sum_{j=0}^{m} a_j z_{k-j}(t) = -\sum_{j=0}^{m} b_j z_{k+j}(t),
\]
\[
k = -m, -m + 1, \ldots, -1.
\]

As the result one arrives at the delay-free system
\[
A \frac{d}{dt} \tilde{z}(t) = B \tilde{z}(t).
\]

According to (10) functions \( z_k(t) \) satisfy the conditions
\[
z_{k+1}(0) = z_k(h), \quad k = -m, -m + 1, 0, 1, \ldots, m - 2,
\]
and the algebraic property (7) which can be written in the terms of the functions \( z_k(\tau) \) as follows

\[
\sum_{j=0}^{m} \sum_{i=0}^{m} a_i b_j z_{j-i-1}(h) = -1/2. \quad \square
\]

4. Lyapunov–Krasovskii functionals

In this section we extend some basic results on the Lyapunov–Krasovskii functionals obtained for the case of retarded type time delay systems to the case of scalar neutral type time delay equations. It is shown how the quadratic Lyapunov functional Eq. (3) can be used for the construction of the complete Lyapunov–Krasovskii functionals. It is desirable to have for the Lyapunov functionals some lower and upper bounds that depend on the state \( x_t \) and not on the time derivative of the state, see [3].

This is why, we propose a modification of the functional (3) which consists in elimination of the time derivative of the state in terms of the functional.

4.1. Transformed functional \( v_0(x_t) \)

As several terms of the functional (3) include explicitly the time derivative of the solution, \( x(t + \theta) \), we are going to transform the functional in such a way that in the new form the functional does not depend explicitly on the time derivative. The transformation consists mainly in a systematic application of the integration by parts to the corresponding terms. To illustrate this, let us consider the term

\[
I = -2a_j a_i u((j - i)h) x(t) + 2a_j a_i u((j - i)h) x(t - ih)
\]

Integrating the term by parts, the following equality for \( I \) can be obtained

\[
I = -2a_j a_i u((j - i)h) x(t) + 2a_j a_i u((j - i)h) x(t - ih)
\]

The new expression for \( I \) does not include the time derivative. Following similar computations one may transform all other terms in order to eliminate the time derivative. As a result the transformed functional looks as

\[
v_0(x_t) = u(0) \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2 + 2 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]
\]

\[
\times \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{-ih}^{0} \left[ u(t + \theta) \right] \left[ \frac{\partial u(t + \theta)}{\partial \theta} \right] x(t + \theta) d\theta
\]

\[
+ \sum_{j=1}^{m} \sum_{i=1}^{m} \int_{-ih}^{0} \left[ x(t + \theta) \right] \left[ \frac{\partial u(t + \theta)}{\partial \theta} \right] x(t + \theta) d\theta
\]

\[
\times \left[ \sum_{j=0}^{m} a_j x(t - jh) \right] \left[ \frac{\partial u(t - jh)}{\partial \theta} \right] x(t - jh) d\theta
\]

\[
\times \Delta u'(i + j - l)h a_j x(t + \theta) (k - l)h d\theta.
\]  

By virtue of the jumps of the first derivative of the Lyapunov function \( u(\tau) \) see Lemma 3.

4.2. Complete type functionals

In this subsection based on the quadratic Lyapunov functional (3) we construct a complete type Lyapunov–Krasovskii functional with a prescribed time derivative which involves not only the simple quadratic form \( x^2(t) \), see [9].

**Theorem 4.** Given the functional

\[
w(x_t) = \sum_{j=0}^{m} \mu_j x^2(t - jh) + \sum_{j=1}^{m} v_j \int_{-jh}^{0} x^2(t + \theta) d\theta,
\]

where \( \mu_j \) and \( v_j \) are non-negative constants. Then the functional

\[
v(x_t) = \left[ \sum_{j=0}^{m} (\mu_j + jh v_j) \right] v_0(x_t)
\]

\[
+ \sum_{j=1}^{m} \int_{-jh}^{0} \left[ (\mu_j + (j + h) \theta) v_j \right] x^2(t + \theta) d\theta,
\]

where \( v_0(x_t) \) is defined by Eq. (11), is such that \( \frac{d}{dt} v(x_t) = -w(x_t) \), for \( t \geq 0 \), along solutions of the Eq. (1).

**Proof.** The statement can be checked by direct calculations of the time derivative. \( \square \)

It is worth to mention that functional (13) is said to be of the complete type if all parameters \( \mu_j \) and \( v_j \) are positive.

Observe that Theorem 4 still be valid when (1) is non exponentially stable, because of the fact that the functional (13) is constructed with the given time derivative (12).

4.3. Quadratic bounds

In this subsection we present some useful bounds for the complete type functionals.

**Theorem 5.** Let Eq. (1) be exponentially stable. If \( \mu_j, j = 0, 1, \ldots, m \) and \( v_j, j = 1, 2, \ldots, m \) are positive constants, then there exist \( \beta_j > 0 \), \( j = 1, 2, 3, 4 \), such that functional (13) satisfies for \( t \geq 0 \) the following inequalities

- \( \beta_1 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2 + \beta_2 \left[ \sum_{j=1}^{m} \int_{-jh}^{0} x^2(t + \theta) d\theta \right] \leq v(x_t) \),
- \( v(x_t) \leq \beta_3 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2 + \beta_4 \left[ \sum_{j=1}^{m} \int_{-jh}^{0} x^2(t + \theta) d\theta \right] \).

**Proof.** Consider the functional

\[
\tilde{v}(x_t) = v(x_t) - \gamma_1 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right] - \gamma_2 \int_{-jh}^{0} x^2(t + \theta) d\theta.
\]

Its time derivative is \( \frac{d}{dt} \tilde{v}(x_t) = -\tilde{w}(x_t) \), where

\[
\tilde{w}(x_t) = w(x_t) + 2\gamma_1 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]
\]

\[
\times \left[ \sum_{k=0}^{m} \int_{-k}^{0} b_k x(t - kh) \right] + \gamma_2 \left[ \sum_{j=1}^{m} \int_{-jh}^{0} x^2(t + \theta) d\theta \right]
\]

\[
\times \left[ x(t) x(t - h) \cdots x(t - mh) \right] L(\gamma_1, \gamma_2)
\]

\[
\times \left[ \sum_{j=1}^{m} v_j \int_{-jh}^{0} x^2(t + \theta) d\theta \right].
\]
Here the matrix \( L \) is given by

\[
L(\gamma_1, \gamma_2) = \text{diag} [\mu_0, \mu_1, \ldots, \mu_m] + \gamma_2 \text{diag} [m, -1, \ldots, -1] + \gamma_1 \begin{bmatrix}
a_0 & a_1 & \cdots & a_m \\
a_1 & a_1 & \cdots & a_m \\
\vdots & \vdots & \ddots & \vdots \\
a_m & a_1 & \cdots & a_m
\end{bmatrix}
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_m
\end{bmatrix}
\]
and
\[
\beta_4 = \left[ \sum_{j=0}^{m} (\mu_j + jv_j) \right] \left[ bu_1 + au_2 + \frac{m(m+1)}{2} h \right] \
\times (b^2 u_1 + 2abu_2 + a^2 u_3) + \frac{m(m+1)}{2} a^2 u_4 + \lambda. \quad \Box
\]

**Remark 2.** It is worth to be mentioned that the exponential stability of Eq. (1) is crucial for the derivation of the lower estimation of the functional \( v(x_t) \) in Theorem 5. Indeed, if the equation is not exponentially stable, then the basic formula for the derivation, formula (14), is not valid any more. As a consequence, there is no chance to obtain positive constants \( \beta_1 \) and \( \beta_2 \) that satisfy the first inequality of the theorem.

**Corollary 1.** Let Eq. (1) be exponentially stable. If \( \mu_j, j = 0, 1, \ldots, m \) and \( v_j, j = 1, 2, \ldots, m \) are positive constants, then the functional (13) satisfies the inequalities
\[
\alpha_1 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2 \leq v(x_t) \leq \alpha_2 |x_t|^2_{l_{\infty}},
\]
where \( \alpha_1 = \beta_1 \) from the previous theorem, and
\[
\alpha_2 = \left[ \sum_{j=0}^{m} (\mu_j + jv_j) \right] \beta + \frac{m(m+1)}{2} h \left( b^2 u_1 + 2abu_2 + a^2 u_3 + a^2 u_4 \right).
\]

5. Application

The objective of this section is to describe a systematic procedure to determine exponential estimates for the solutions of the multiple delays scalar neutral equations. The proof uses the quadratic Lyapunov–Krasovskii functionals for the exponentially stable linear neutral Eq. (1).

5.1. Exponential estimates

**Lemma 4.** Let Eq. (1) be exponentially stable. Given positive constants \( \mu_j, j = 0, 1, 2, \ldots, m \) and \( v_j, j = 1, 2, \ldots, m \). Then there exists \( \alpha_1 > 0 \) such that functional (13) along solutions of Eq. (1) satisfies the following inequality
\[
\frac{d}{dt} v(x_t) + 2\alpha_1 v(x_t) \leq 0, \quad t \geq 0.
\]

**Proof.** Observe first that by Theorem 5
\[
\frac{d}{dt} v(x_t) + 2\alpha_1 v(x_t) \leq -2\alpha_1 v(x_t)
\]
\[
+ 2\alpha_1 \left[ \beta_3 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2 + \beta_4 \sum_{j=1}^{m} x^2(t + \theta)d\theta \right] \leq -\sum_{j=0}^{m} \mu_j x^2(t - jh) + 2\alpha_1 \beta_3 \left[ \sum_{j=0}^{m} a_j x(t - jh) \right]^2
\]
\[
- \sum_{j=1}^{m} \int_{-j}^{0} [v_j - 2\sigma_1 \beta_4] x^2(t + \theta) d\theta \leq -[x(t) x(t - h) \cdots x(t - mh)] L(\sigma_1) \left[ \begin{array}{c} x(t) \\ x(t - h) \\ \vdots \\ x(t - mh) \end{array} \right]
\]
\[
- \sum_{j=1}^{m} [v_j - 2\sigma_1 \beta_4] \int_{-j}^{0} x^2(t + \theta) d\theta,
\]
where the matrix
\[
L(\sigma_1) = \text{diag}[\mu_0, \mu_1, \ldots, \mu_m] + 2\alpha_1 \beta_3 \left[ \begin{array}{ccc} a_0 & a_1 & \cdots & a_m \end{array} \right],
\]
and
\[
\mu = \sqrt{\alpha_2/\alpha_1}, \quad \text{with } \alpha_2, \alpha_1 \text{ defined in Corollary 1 and } \sigma_1 \text{ defined in Lemma 4}.
\]

**Corollary 2.** The solution of Eq. (1) for the initial condition \( \varphi \) satisfies the following exponential estimate
\[
\sum_{j=0}^{m} a_j x(t - jh, \varphi) \leq \mu \| \varphi \|_{l_{\infty}} e^{-\sigma_1 t}, \quad t \geq 0,
\]
where \( \mu = \sqrt{\alpha_2/\alpha_1} \), with \( \alpha_2, \alpha_1 \) defined in Corollary 1 and \( \sigma_1 \) defined in Lemma 4.

**Lemma 5.** Let polynomial
\[
p(s) = s^m + \sum_{j=1}^{m} a_j s^{m-j}
\]
be Schur stable. Then there exist \( \kappa \geq 1 \) and \( \sigma_2 > 0 \) such that
\[
\| S^k \| \leq \kappa e^{-\kappa \sigma_2 t}, \quad k = 0, 1, 2, \ldots,
\]
where matrix \( S \) has been introduced in Lemma 3.

**Lemma 6** ([8]). Let polynomial \( p(s) \) be Schur stable. Consider the difference equation
\[
x(t) + \sum_{j=1}^{m} a_j x(t - jh) = f(t), \quad t \geq 0,
\]
where \( |f(t)| \leq \mu \| \varphi \|_{l_{\infty}} e^{-\sigma_1 t}, t \geq 0 \). Then, the solution \( x(t, \varphi) \) of (15) satisfies the inequality
\[
|x(t, \varphi)| \leq \gamma \| \varphi \|_{l_{\infty}} e^{-\sigma_2 t}, \quad t \geq 0,
\]
where \( \gamma = \kappa \left[ 1 + \mu + \frac{\sigma_2}{\sigma_1} \right] \) and \( \sigma = \sigma_0 - \epsilon \). Here \( \epsilon \in (0, \sigma_0) \), \( \sigma_0 = \min(\alpha_1, \alpha_2) \), and \( \mu = \sqrt{\alpha_2/\alpha_1} \).

Observe that the parameters \( \alpha_1, \alpha_2 \) and \( \sigma_1 \) depend on the positive constants \( \mu_j \) and \( v_j, j = 1, 2, \ldots, m \) and on the quantities \( u_1, u_2, u_3 \) and \( u_4 \). Then, the constants \( \gamma \) and \( \sigma \) depend on \( \mu_j \) and \( v_j \) and on the Lyapunov function.

We are now able to state the main result of the section, the exponential estimates for solutions of Eq. (1).
Theorem 6. Let Eq. (1) be exponentially stable. Given positive constants $\mu_j, j = 0, 1, 2, \ldots, m$ and $\nu_j, j = 1, 2, \ldots, m$, then the solutions of Eq. (1) satisfy the inequality

$$|x(t, \varphi)| \leq \gamma |\varphi|_{\text{im}} e^{-\sigma t}, \quad t \geq 0,$$

where $\gamma$ and $\sigma$ were stated in Lemma 6.

6. Example

Consider the scalar equation

$$\dot{x}(t) - 0.1x(t - 1) - 0.2x(t - 2)$$

$$= -x(t) - 0.3x(t - 1) - 0.4x(t - 2).$$

(16)

The equation is exponentially stable. In Fig. 1 we plot the Lyapunov function, where we can observe the symmetry of $u(\tau)$, we also can see that although $u(\tau)$ is continuous it is not smooth because it presents removable discontinuities at the points $\tau = -1, \tau = 0$, and $\tau = 1$.

The removable discontinuities of the function $u(\tau)$ imply that the first derivative of the Lyapunov function presents jump discontinuities at the same points, see Fig. 2, notice that $u'(\tau)$ is not continuous because of the jumps, we call this jumps as $\Delta u'(-1), \Delta u'(0)$, and $\Delta u'(1)$.

Finally, in Fig. 3 we present the second derivative of the Lyapunov function $u'(\tau)$, notice that at the points $\tau = -1, \tau = 0$, and $\tau = 1$, the second derivative takes the form $\Delta u'(-1)\delta(\tau + 1), \Delta u'(0)\delta(\tau)$, and $\Delta u'(1)\delta(\tau - 1)$, respectively, where $\delta(\tau)$ is the Dirac function.

Consider Eq. (16). Let us select $\mu_0 = 1.5, \mu_1 = \mu_2 = \nu_1 = \nu_2 = 1$. Applying Theorem 5 to Eq. (16) we arrive at the following bounds for the functional (13)

$$0.7307 \left | x(t) - 0.1x(t - 1) - 0.2x(t - 2) \right |^2$$

$$+ 0.4829 \left [ \int_{-1}^{0} \chi^2(t + \theta) d\theta + \int_{-2}^{0} \chi^2(t + \theta) d\theta \right ] \leq v(x_1),$$

$$v(x_1) \leq 16.5143 \left | x(t) - 0.1x(t - 1) - 0.2x(t - 2) \right |^2$$

$$+ 23.4979 \left [ \int_{-1}^{0} \chi^2(t + \theta) d\theta + \int_{-2}^{0} \chi^2(t + \theta) d\theta \right ].$$

The value $\sigma_1$ from Lemma 4 is equal to 0.0144, while $\mu$ from Corollary 2 is equal to 11.2973. For matrix $S$ the inequality $\|S\| \leq 2e^{-0.6\beta\delta}$ holds for $k = 0, 1, 2, \ldots$, so we may select $\kappa = 2$ and $\sigma_2 = 0.6$, see Lemma 5. Now, the value $\sigma_0$ from Lemma 6 is equal to 0.0144, and if we assume that $\epsilon = 0.0004$, then $\sigma = 0.014$. The value $\gamma$ from Theorem 6 is equal to 2.0803 $\times 10^9$ and we arrive at the following exponential estimate for the solution of Eq. (16)

$$|x(t, \varphi)| \leq 2.0803 \times 10^9 |\varphi|_{\text{im}} e^{-0.014t}, \quad t \geq 0.$$

7. Concluding remarks

In this paper, we present a new explicit expression for the complete type Lyapunov–Krasovskii functionals, and provide exponential estimates for solutions of a scalar neutral type time delay equation. It is shown that in the case of exponentially stable equations the functionals admit lower and upper quadratic bounds. A new definition of the Lyapunov function which does not demand exponential stability of the time delay equation is also given. All principal results are illustrated by an example.

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References


