Robust control design through the attractive ellipsoid technique for a class of linear stochastic models with multiplicative and additive noises

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This paper concerns the robust ‘practical’ stabilization for a class of linear controlled stochastic differential equations subject to both multiplicative and additive stochastic noises. Sufficient conditions of the stabilization are provided in two senses. In the first sense, it is proven that almost all trajectories of the stochastic model converge in a ‘mean-square sense’ to a bounded zone located in an ellipsoidal set, while the second one ensures the convergence to a zero zone in probability one. The considered control law is a linear state feedback. The stabilization problem is converted into the corresponding attractive averaged ellipsoid ‘minimization’ under some constraints of bilinear matrix inequalities (BMIs) type. Some variables permit to represent the BMIs problem in terms of linear matrix inequalities (LMIs) problem, which are resolved in a straight manner, using the conventional LMI-MATLAB toolbox. Finally, the numerical solutions of a benchmark example and a practical example are presented to show the efficiency of the proposed methodology.

Keywords: stochastic differential equations; linear matrix inequalities; attractive ellipsoid method.

1. Introduction

Some real-world phenomena present disturbances and perturbations of non-deterministic nature. The disturbances deal with additive noises, such as measurement noises in electrical systems (Gray et al., 1999), daily weather effects (Richardson, 1981), variations in population dynamics (Sykes, 1969), among others. On the other hand, some economic processes (e.g. El Karoui et al., 1998; Shreve, 2004), optoelectronic devices (Konstantatos et al., 2006) and many other physical systems are affected by multiplicative disturbances of stochastic nature. Sometimes, these effects arise just in additive (multiplicative) form, which reduce the mathematical analysis of the systems. However, in the case of some economic pricing options (Anteneodo & Riera, 2005), it is necessary to analyse the stability and the synthesis of robust control strategies in dynamics affected by both additive and multiplicative stochastic noises.

This class of models (stochastic differential equations) is governed by its behaviour properties which can be analysed by two main techniques: the quantitative techniques and the qualitative ones. ‘Quantitative methods’ (Appleby & Flynn, 2006; El Bouhtouri & El Hadri, 2003) need an exact solution (closed-form expression) of the equations, which, for most of the systems, specially non-linear, is rarely possible. These methods rely on general known properties of the systems, as well as boundary conditions, which are closely related to numerical approximations. The main advantage of these methods is that they provide a specific information of the solution at any time. On the other hand, ‘qualitative
methods’ permit to investigate the general behaviour of the solutions, i.e., properties such as boundedness, periodicity, convergence, stability and so on, without the need of having an explicit form of the solution. Examples of these methods are the Lyapunov’s second method, comparison theorems, among others (see, e.g. Arnold & Schmalfuss, 2001).

Concerning linear controlled stochastic differential equations (LCSDEs), one of the main qualitative properties is the, so-called, ‘robust practical stability’. Some quantitative methodologies have been proposed to synthesize the control schemes. In Poznyak et al. (2002), a robust stochastic maximum principle is proven with the aid of deterministic min–max tools and the stochastic maximum principle. The well-known work of Zhou (1991) deals with an optimal stochastic control problem, where the diffusion coefficient also depends on the control, which can change its diffusion structural properties. In this problem, the maximum principle, dynamic programming and their connections are established within a unified framework of viscosity solution. The $H^\infty$ approach is considered in Ugrinovskii (1998), where a state feedback control is proposed for a class of systems affected by uncertain multiplicative white noise perturbations, satisfying a certain variance constraint.

The aforementioned methods can achieve excellent results in the presence of a certain minimal knowledge of the system. However, when there is a lack of information, these methods may not obtain desirable results. The robust attractive ellipsoid methods (Bertsekas, 1994; Kurzhanskii & Valyi, 1997; Schweppe, 1973) come up with a qualitative approach which allows us to analyse and synthesize control schemes for a class of uncertain systems. This method is based on the second Lyapunov method and the concept of invariant sets. The solutions are expressed as an optimization problem restricted to ‘bilinear matrix inequalities’ (BMIs) (see Safonov et al., 1994) and considering an effective numerical method for the design of the corresponding feedback control (Kurzhanskii & Valyi, 1997). The comprehensive survey on invariant sets can be found in Blanchini & Miani (2007). In Polyak et al. (2006), the problem of synthesis of a static state feedback controller for a linear time-invariant system, minimizing the size of the corresponding invariant ellipsoid, was reduced to optimization of a linear function under some set of ‘linear matrix inequality’ (LMI) application constraints (Nagapal et al., 1994; Abedor et al., 1996; Safonov et al., 1994).

In this article, we deal with the problem of robust practical stabilization (a zone-convergence analysis) of an LCSDE with multiplicative and additive noises, by means of attractive ellipsoid. Since the class of stochastic equations to analyse contains a Wiener process as an external noise, the Ito’s integral is used as the main tool to obtain a stabilizing robust control law.

1.1 Main constraints and contribution

The main ‘constraints’ accepted in this paper are as follows:

- we consider the class of linear stochastic differential equations with diffusion terms linearly dependent on the state and control;
- as the control action, we use static linear feedback control.

The principal ‘contributions’ are as follows:

- A specific form of BMI (which by a special transformation is shown to be converted into an LMI) is obtained which provides the mean-square convergence into a bounded ellipsoidal set and, as a result, guarantees the robust ‘practical stability’ property of the designed controller for a wide class of stochastic linear systems with controlled diffusion terms.
- Sufficient conditions to exponential convergence to a zero zone.

- The numerical matrix optimization procedure, based on the ‘interior point method’ (Nesterov & Nemirovsky, 1994), which under the obtained LMI constraints provides the optimal numerical values of the control $K$ guaranteeing a ‘minimal’ attractive ellipsoid.

1.2 Outline of the paper

In the next section, some necessary concepts, definitions and notation to present the formulation problem are introduced. Besides, the stochastic model description and the problem formulation are given. On the other hand, there are given some important propositions to present the main results which describe the robust attractive ellipsoid where all trajectories of the closed-loop system converge in mean square and with the probability one. Then, in Section 3, the ideas for converting the BMI problem to LMI problem are shown. In Section 4, the numerical illustrative results for a benchmark example and actual example are presented in different cases. Finally, some conclusions are given.

2. System description and problem formulation

We consider $\Omega$ a non-empty set called ‘sample space’, $\mathcal{F} \subseteq 2^\Omega$ a $\sigma$-algebra called ‘event space’, $P$ a probability measure and $t \in [0, T] \subseteq \mathbb{R}_+$. In what follows, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ denotes a filtered probability space with $\{W_t\}_{t \geq 0}$ being an $m$-dimensional $(W_t = W^1_t, \ldots, W^m_t)^\top$ standard Brownian motion, $m \in \mathbb{N}$. We suppose the filtration satisfies the usual hypotheses.

Consider the following stochastic differential equation:

$$\begin{align*}
\mathrm{d}x_t &= (Ax_t + Bu_t + b)\mathrm{d}t + \sum_{i=1}^{m}(C_ix_t + D_iu_t + \sigma_i)dW^i_t, \\
x_t, b, \sigma_i &\in \mathbb{R}^n; \quad A, C_i \in \mathbb{R}^{n \times n}; \quad B, D_i \in \mathbb{R}^{n \times k}
\end{align*}$$

(2.1)

The variables of this equation have the following ‘physical’ interpretation:

- $x_t \in \mathbb{R}^n$ is the vector of the states of the model (2.1) at time $t \geq t_0 \geq 0$;

- $u_t \in \mathbb{R}^k$ is the vector of the control action (at time $t$) which should be suggested by a designer to stabilize this system in the origin;

- $W^i_t (i \in \{1, \ldots, m\})$ is a standard scalar Brownian motion characterizing external random perturbations to the vector-state dynamics satisfying (2.1); in engineering applications, it is referred as a ‘white noise’ external perturbation affecting the given model dynamics;

- the constant matrices $A, C_i \in \mathbb{R}^{n \times n}, B, D_i \in \mathbb{R}^{n \times k}$ and the constant scalars $b$ and $\sigma_i$ are associated with the parameters of the considered models and are supposed below to be a priori known.

Remark 2.1 Note that in most of the publications there are studied only the case $C_i = 0, D_i = 0 (i = 1, \ldots, m)$ corresponding to the, so-called, ‘additive noise’ presence which keeps the same ‘power signal’ ($\sigma_i = \text{const}, \neq 0$) independently of the closeness of the current state $x_t$ to the origin. When $\sigma_i = 0$ for all $i$ and at least one of $C_i \neq 0$ (but $D_i = 0 (i = 1, \ldots, m)$), we deal with the, so-called, ‘multiplicative noise’ case which reflects the noise-decreasing (vanishing) property of some models.
been considered closer to the point $x = 0$. If at least one $D_i \neq 0$, the corresponding dynamics (2.1) has the controllable diffusion term reflecting the fact that in this situation the control action $u_t$ may also affect the noise power.

Consider below two ‘practical’ motivating example models studied in earlier publications.

**EXAMPLE 2.1** Consider a gene regulatory network, which is a collection of DNA segments that interact with each other through their messenger RNA (mRNA) and some protein expression products. The conditions that govern the genetic expression of some proteins permit to model this class of systems as stochastic models. Under certain conditions, the variations may produce genetic disorders, leading to medical diseases. Controlling the mRNA is a gene therapy alternative, which allows us to modify the mRNA contents. This action helps to prevent diseases or to produce certain types of proteins.

The model of a gene regulatory network can be represented by a linear stochastic model of the form (see Ozbudak et al., 2002; El Samad & Khammash, 2004; Chunguang et al., 2006)

$$dx_t = Ax_t \, dt + \Xi x_t \, dW_t$$

$$x_t \in \mathbb{R}^2, \quad A \in \mathbb{R}^{2 \times 2}, \quad \Xi \in \mathbb{R}^{2 \times 2},$$

(2.2)

where

- $x_{1,t}$ is the number of mRNA molecules;
- $x_{2,t}$ is the number of protein molecules. The states are defined in terms of an equilibrium point;
- $A = \begin{bmatrix} -\gamma_R & -K_2 \\ K_P & -\gamma_P \end{bmatrix},$

where $\gamma_R$ and $\gamma_P$ represent the decay rates of mRNA and protein, respectively, $K_P$ is the transcription rate and $K_R$ is the translation rate, whose form is set to be $K_R = K_2 - K_1 P_0$, where $K_1, K_2 \in \mathbb{R}$, and $P_0$ the end proteins product, according to El Samad & Khammash (2004).

- $\Xi$ represents the constant matrix which corrupts the parameters $\gamma_R$, $\gamma_P$, $K_P$, and $K_R$.

**EXAMPLE 2.2** (The reinsurance–dividend management; Taksar & Zhou, 1998) Consider the following $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $\mathbb{R}$-valued random processes

$$dy(t) = [a(t)\tilde{\mu}^\alpha - \delta^\alpha - c(t)] \, dt - a(t)\sigma^\alpha \, dW(t),$$

$$y(0) = y_0,$$

where

- $y(t)$ is the value of the liquid assets of a company at time $t$,
- $c(t)$ is the dividend rate paid out to the shareholder at time $t$,
- $\tilde{\mu}^\alpha$ is a difference between premium rate and expected payment on claims per unit time (‘safety loading’),
- $\delta^a$ is the rate of the debt repayment,
- $[1 - a(t)]$ is the reinsurance fraction,
- $\sigma^a := \sqrt{\lambda^a E[\eta^2]}$ ($\lambda^a$ is the intensity of Poisson process, $\eta$ is the size of claim).

Here, the controls are

$$u_1(t) := a(t) \in [0, 1] \quad \text{and} \quad u_2(t) := c(t) \in [0, c^+]$$

Certainly, this example corresponds the situation with a controllable diffusion term.

Below, we will consider the linear feedback

$$u_t := K x_t, \quad K \in \mathbb{R}^{k \times n}, \quad (2.3)$$

and study the capacity of such feedback to stabilize the considered system in some probabilistic sense. Substituting (2.3) into (2.1) leads to

$$dx_t = [(A + BK)x_t + b]dt + \sum_{i=1}^{m} [(C_i + D_i K)x_t + \sigma_i]dW_t^i. \quad (2.4)$$

Let $P$ be a positive definite matrix in $\mathbb{R}^{n \times n}$ satisfying

$$0 < \beta I_{n \times n} \leq P \leq \alpha I_{n \times n}.$$ 

Define also the quadratic function

$$V(x) := x^T P x$$

for which at the trajectories $x = x_t$ of the system (2.4) there exists (under the accepted assumptions) the mathematical expectation

$$\mathbb{E} = E\{V(x_t)\} \quad (2.5)$$

(which is, obviously, differentiable if we apply the Ito’s calculus to $dV(x_t)$). Now we are ready to formulate the first main result.

Introduce the extended vector

$$z_t := (x_t^T \quad b^T \quad \sigma_1^T \quad \cdots \quad \sigma_m^T)^T \in \mathbb{R}^{n(m+2)}. \quad (2.6)$$

**Lemma 2.1** Under the accepted assumptions for any non-negative $\alpha$, $\varepsilon$ and $\kappa$, the following representation holds:

$$\frac{d}{dt} V(t) = E\{z_t^T W z_t\} - \alpha V(t) + \beta, \quad (2.7)$$
where

\[
W := \begin{bmatrix}
\tilde{W}_{11} & W_{12} & W_{13} & \cdots & W_{1,m+2} \\
W_{21} & W_{22} & 0 & \cdots & 0 \\
W_{31} & 0 & W_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
W_{m+2,1} & 0 & 0 & \cdots & W_{m+2,m+2}
\end{bmatrix},
\quad (2.8)
\]

\[
\tilde{W}_{11} := P \left( A + BK + \frac{\alpha}{2} I_{n \times n} \right) + \left( A + BK + \frac{\alpha}{2} I_{n \times n} \right)^\top P
\]

\[
+ \sum_{i=1}^{m} [(C_i + D_i K)^\top P(C_i + D_i K)] \in R^{n \times n},
\]

\[
W_{12} = W_{21}^\top := P, \quad W_{22} := -\varepsilon I_{n \times n}, \quad W_{j,j} := P - \kappa I_{n \times n},
\]

\[
W_{j1} = W_{1j}^\top := P(C_{j-2} + D_{j-2} K), \quad j \in \{3, \ldots, m + 2\},
\]

and

\[
\beta := \varepsilon \|b\|^2 + \kappa \sum_{i=1}^{m} \|\sigma_i\|^2.
\quad (2.9)
\]

**Proof.** From the multidimensional Itô’s formula (see, e.g. Poznyak, 2009), we derive

\[
dV(x_t) = \frac{\partial V}{\partial t}(x_t)dt + \left[ \frac{\partial V}{\partial x}(x_t) \right]^\top dx_t
\]

\[
+ \frac{1}{2} \text{tr} \left\{ \left[ \frac{\partial^2 V}{\partial x^2}(x_t) \right]^\top \sum_{i=1}^{m} [(C_i + D_i K)x_t + \sigma_i][(C_i + D_i K)x_t + \sigma_i]^\top \right\} dt.
\]

Replacing \(dx_t\) from (2.4) and taking into account that

\[
\frac{\partial V}{\partial t}(x_t) \equiv 0, \quad \frac{\partial V}{\partial x}(x_t) = 2x^\top P, \quad \frac{\partial^2 V}{\partial x^2}(x_t) = 2P,
\]

we obtain in the integral form

\[
V(x_t) - V(x_0) = \int_{t_0}^{t} 2x_s^\top P[(A + BK)x_s + b]ds + \int_{t_0}^{t} \sum_{i=1}^{m} 2x_s^\top P[(C_i + D_i K)x_s + \sigma_i]dW_s^i
\]

\[
+ \int_{t_0}^{t} \sum_{i=1}^{m} \text{tr}\{P[(C_i + D_i K)x_s + \sigma_i][(C_i + D_i K)x_s + \sigma_i]^\top\}ds.
\]

Taking mathematical expectation of both parts and in view of the local properties of Itô’s integral, we conclude that

\[
E \left\{ \int_{t_0}^{t} 2x_s^\top P[(C_i + D_i K)x_s + \sigma_i]dW_s^i \right\} = 0
\]
Proof. It follows directly from (2.7) if we take into account that
\( (2.10) \) as the ordinary differential equation (2.7).

By the property \( \text{tr}\{AC\} = \text{tr}\{CA\} \), the last equality can be represented as
\[
E[V(x_t)] - E[V(x_0)] = \int_{t_0}^{t} 2E[x_s^T P[(A + BK)x_s + b]] ds + \sum_{i=1}^{m} \int_{t_0}^{t} \text{tr}[E[P((C_i + D_iK)x_s + \sigma_i)[(C_i + D_iK)x_s + \sigma_i]^T]] ds.
\]

Calculating the derivative of both sides of the last identity, we obtain
\[
\frac{d}{dt} E[V(x_t)] = E[2x_t^T P[(A + BK)x_t + b]] + \sum_{i=1}^{m} \text{tr}[E[((C_i + D_iK)x_t + \sigma_i)[(C_i + D_iK)x_t + \sigma_i]^T] P[(C_i + D_iK)x_t + \sigma_i]].
\]

Using the definition (2.5) and the property \( 2x^T A x = x^T (A + A^T) x \), we are able to rewrite the last equation as
\[
\frac{d}{dt} V(t) = E[x_t^T \{ P(A + BK) + (A + BK)^T P + \} x_t] + E \left\{ x_t^T \sum_{i=1}^{m} [(C_i + D_iK)^T P(C_i + D_iK)] x_t \right\} + E[x_t^T P b] + E \left\{ x_t^T \sum_{i=1}^{m} [(C_i + D_iK)^T P \sigma_i] \right\} + E \left\{ \sum_{i=1}^{m} [\sigma_i^T P(C_i + D_iK)] x_t \right\} + \sum_{i=1}^{m} \sigma_i^T P \sigma_i,
\]
and adding and subtracting the terms \([\pm \alpha \hat{V}(x_t)]\), \([\pm \epsilon \|b\|^2]\) and \([\pm \sum_{i=1}^{m} \kappa \|\sigma_i\|^2]\), one can represent (2.10) as the ordinary differential equation (2.7).

The following result is the consequence of the above lemma.

Corollary 2.1 For any matrices \( P, K \) and constants \( \alpha, \beta \) and \( \kappa \) such that \( W < 0 \) (2.8), the following differential inequality holds:
\[
\frac{d}{dt} V(t) \leq -\alpha V(t) + \beta
\]
so that
\[
V(t) \leq e^{-\alpha(t-t_0)} V(t_0) + \frac{\beta}{\alpha} (1 + e^{-\alpha(t-t_0)}) = \frac{\beta}{\alpha} + O(e^{-\alpha(t-t_0)}).
\]

Proof. It follows directly from (2.7) if we take into account that \( W < 0 \).
By the previous corollary, we obtain the result that describes the sufficient conditions of the state stabilization.

**Theorem 2.1** If there exists a real positive number \( \tau < \beta \) and the collection \( \{ K, P, \alpha, \beta, \tau \} \) satisfying the following matrix inequality:

\[
\mathcal{Y} := W - \tau \frac{\alpha}{\beta} \tilde{I}^\top P \tilde{I} \leq 0
\]

with

\[
\tilde{I} = (I_{n \times n}, O_{n \times n}, \ldots, O_{n \times n}) \in \mathbb{R}^{n \times n(m+2)},
\]

then the best parameters minimizing the convergence attractive ellipsoid are

\[
\{ K^*, P^*, \alpha^*, \beta^*, \tau^* \} = \text{Arg sup}_{\alpha > 0, \beta > 0, \tau > 0, \mathcal{Y} < 0} \text{tr} \left\{ \frac{\alpha}{\beta - \tau} P \right\}.
\]

**Proof.**

From (2.11), (2.8) and (2.6), define

\[
A_0 := W \in \mathbb{R}^{n(m+2) \times n(m+2)},
\]

\[
A_1 := \begin{bmatrix}
-\frac{\alpha}{\beta} P & O_{n \times n} & \cdots & O_{n \times n} \\
O_{n \times n} & O_{n \times n} & \cdots & O_{n \times n} \\
\vdots & \vdots & \ddots & \vdots \\
O_{n \times n} & O_{n \times n} & \cdots & O_{n \times n}
\end{bmatrix} \in \mathbb{R}^{n(m+2) \times n(m+2)}
\]

and

\[
f_i(z_t) := z_t A_i z_t, \quad i = 1, 2,
\]

where

\[
x_t = I_{n \times n} z_t.
\]

By \( S \)-procedure (see 12.3.2, Poznyak, 2008), the inequality

\[
f_1(z_t) < -1
\]

implies that

\[
f_0(z_t) < 0
\]

if only if there exists \( \tau \geq 0 \) such that

\[
A_0 - \tau A_1 < 0
\]

given by (2.13). Since the attractive ellipsoid is given by

\[
\limsup_{t \to \infty} \mathcal{V}(t) = \limsup_{t \to \infty} E\{x_t^\top P x_t\},
\]

then the maximum of \( P \) (in fact, its trace) implies a smaller dynamics \( x_t \).

\[ \square \]
The previous result deals with the convergence to zero zone (ellipsoid zone) in mean-square sense. The next corollary shows that the trajectories of the closed-loop system also converge to the same ellipsoid with probability one.

**Corollary 2.2** Under the conditions of Theorem 2.1, we may guarantee that

\[ P \left\{ \omega : \lim_{t \to \infty} d\left(x_t, S\left(0, \frac{\alpha}{\beta - \tau} P\right)\right) = 0 \right\} = 1, \]  

(2.14)

where

\[ d(x, S) := \inf_{y \in S} \|x - y\|, \]

\[ S(0, \tilde{P}) := \{x \in \mathbb{R}^n : x^\top \tilde{P} x \leq 1\}. \]

**Proof.** Let \( \{t_k\}_{k=0,1,2,...} \) be any monotonically increasing sequence from \( \mathbb{R}_+ \) \( \left(t_k < t_{k+1}, t_k \to \infty\right) \). Then, by the Borel–Cantelli lemma

\[ \sum_{k=0}^{\infty} \chi \left\{ \left[ V(x_{t_k}) - \frac{\beta}{\alpha} \right]_+ \geq \varepsilon > 0 \right\} \overset{a.s.}{\to} \infty, \]  

(2.15)

But, by the generalized Chebyshev inequality (Poznyak, 2009) and in view of (2.12), it follows that

\[ \sum_{k=0}^{\infty} P \left\{ \left[ V(x_{t_k}) - \frac{\beta}{\alpha} \right]_+ \geq \varepsilon > 0 \right\} \leq \varepsilon^{-1} \sum_{k=0}^{\infty} E \left\{ \left[ V(x_{t_k}) - \frac{\beta}{\alpha} \right]_+ \right\} \]

\[ \leq \varepsilon^{-1} \sum_{k=0}^{\infty} E \left\{ e^{-\alpha(t_k-t_0)} \left[ \tilde{V}(x_{t_0}) + \frac{\beta}{\alpha} \right]_+ \right\} \]

\[ = \varepsilon^{-1} \left[ \tilde{V}(x_{t_0}) + \frac{\beta}{\alpha} \right] \sum_{k=0}^{\infty} e^{-\alpha(t_k-t_0)} < \infty. \]

The convergence of the series (2.15) with probability one for any small \( \varepsilon > 0 \) and any sequence \( \{t_k\}_{k=0,1,2,...} \) exactly means that for almost all \( \omega \in \Omega \) and any sequence \( \{t_k\}_{k=0,1,2,...} \), there exists a random (but finite) number \( k^+(\omega) \overset{a.s.}{\to} \infty \) such that for all \( k \geq k^+(\omega) \) we have \( \chi \left\{ \left[ V(x_{t_k}) - \frac{\beta}{\alpha} \right]_+ \geq \varepsilon > 0 \right\} \overset{a.s.}{=} 0 \). If this is true for any sequence \( \{t_k\}_{k=0,1,2,...} \), it means that \( \left[ V(x_t) - \frac{\beta}{\alpha} \right]_+ \overset{a.s.}{\to} 0 \), whereas \( t \to \infty \). Corollary is proven.

**Corollary 2.3** If \( \beta = 0 \), i.e. in (2.9) \( \|b\| = \|\sigma_i\| = 0 \) \( (i = 1, \ldots, m) \), and \( \tau = 0 \) in Theorem 2.1, we have global stability with probability one of the origin \( x = 0 \) of the closed-loop system (2.4) containing the diffusion term depending linearly on state and control (2.3).
3. LMI representation of the optimization problem

As it follows from (2.4), if one wishes to ‘maximize’ $P$ by the matrix $K$ and other scalar parameters in some ‘matrix’ sense, then the matrix gain $K$ should be found as the solution of

$$\frac{\alpha}{\beta - \tau} \text{tr} P \rightarrow \sup_{K, \varepsilon, \kappa, \alpha, \tau} \text{tr} P, \quad \tau < 0.$$  
(3.1)

From Theorem 2.1, we define $\Lambda = -\Upsilon > 0$

$$\Lambda = \begin{bmatrix}
-\tilde{W}(2) & A_{12} & A_{13} & \cdots & A_{1,m+2} \\
A_{21} & A_{22} & 0 & \cdots & 0 \\
A_{31} & 0 & A_{3,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{m+2,1} & 0 & 0 & \cdots & A_{m+2,m+2}
\end{bmatrix},$$

$$\tilde{W}(2) := \begin{bmatrix}
P \left( A + \alpha \frac{1}{2} \left( 1 + \frac{\tau}{\beta} \right) n \times n + BK \right) + \left( A + \alpha \frac{1}{2} \left( 1 + \frac{\tau}{\beta} \right) n \times n + BK \right)^\top P \\
+ \sum_{i=1}^{m} (C_i + D_i K)^\top P (C_i + D_i K)
\end{bmatrix},$$

$$A_{12} = A_{21}^\top := -W_{12}, \quad A_{22} := -W_{22}, \quad A_{j,j} := -W_{j,j},$$

$$A_{j1} = A_{1j}^\top := -W_{1j}, \quad j \in \{3, \ldots, m + 2\}.$$ 

Note that the matrix $\Lambda$ is not linear with respect to our variables even for fixed scalar parameters. The principal problem is represented in the expression $(C_i + D_i K)^\top P (C_i + D_i K)$. Let $Q_{1,i}$ be a non-negative matrix such that $(C_i + D_i K)^\top P ((C_i + D_i K)^\top) \preceq Q_{1,i}$ for $i \in \{1, \ldots, m\}$. By the application of the Schur’s complement, we have

$$\begin{bmatrix}
Q_{1,i} & (C_i + D_i K)^\top \\
(C_i + D_i K) & P^{-1}
\end{bmatrix} \succeq 0.$$  
(3.2)

Multiplying (3.2) by two positive matrices, the equivalent inequality is

$$\begin{bmatrix}
I_{n \times n} & 0 \\
0 & P
\end{bmatrix} \begin{bmatrix}
Q_{1,i} & (C_i + D_i K)^\top \\
(C_i + D_i K) & P^{-1}
\end{bmatrix} \begin{bmatrix}
I_{n \times n} & 0 \\
0 & P
\end{bmatrix} \succeq 0,$$

$$\begin{bmatrix}
Q_{1,i} & C_i^\top P + (D_i K)^\top P \\
PC_i + PD_i K & P
\end{bmatrix} \succeq 0$$  
(3.3)

for $i \in \{1, \ldots, m\}$. With similar arguments, we analyse $P (C_i + D_i K)$. For $PBK$, consider $\theta_2 = PBK$. Thus,

$$\hat{W} := \begin{bmatrix}
\tilde{W}(3) & \tilde{W}_{12} & \tilde{W}_{13} & \cdots & \tilde{W}_{1m+2} \\
\tilde{W}_{21} & \tilde{W}_{22} & 0 & \cdots & 0 \\
\tilde{W}_{31} & 0 & \tilde{W}_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{W}_{m+2,1} & 0 & 0 & \cdots & \tilde{W}_{m+2,m+2}
\end{bmatrix} \succeq 0.$$  
(3.4)
and
\[
\tilde{W}_{(3)} := \begin{bmatrix}
P \left( A + \frac{\alpha}{2} \left( 1 + \frac{\tau}{\beta} \right) I_{n \times n} \right) + \left( A + \frac{\alpha}{2} \left( 1 + \frac{\tau}{\beta} \right) I_{n \times n} \right)^\top P \\
+ \theta_2 + \theta_2^\top + \sum_{i=1}^{m} Q_{1,i},
\end{bmatrix}
\]
\[
\tilde{W}_{12} = \tilde{W}_{21} := P, \quad \tilde{W}_{22} := -\varepsilon I_{n \times n}, \quad \tilde{W}_{j,i} := P - \kappa I_{n \times n},
\]
\[
\tilde{W}_{j1} = \tilde{W}_{1j} := PC_j + \theta_{1,j}, \quad j \in \{3, \ldots, m + 2\}.
\]

For solving these LMIs (3.2), (3.3) and (3.4), first we fix the scalar parameters \(\alpha, \kappa, \beta, \tau\) and \(\varepsilon\) and solve our problem with respect to the matrix variables which satisfy the LMI constraints. And second, for the found matrix variables \(\theta_{1,i}, \theta_2\) and \(Q_{1,i}\) for \(i \in \{1, \ldots, m\}\), we solve our problem with respect to the scalar parameters \(\alpha, \kappa, \beta, \tau\) and \(\varepsilon\).

Finally, we find the solution \(\alpha^*, \kappa^*, \beta^*, \tau^*, \varepsilon^*, \theta_{1,i}^*, \theta_2^*\) and \(Q_{1,i}^*\) for \(i \in \{1, \ldots, m\}\). We find \(P^*\) and \(K^*\) from (3.3). This problem can be solved using the MATLAB, LMI toolbox, SeDuMi or Yalmip.

4. Numerical example

Below, we present some numerical simulations to confirm the effectiveness of the proposed stabilization strategy. Consider a stochastic system of the form (2.4) and a proportional control law as in (2.3). The block diagram of the control approach is depicted in Fig.1. We consider two examples: benchmark and gene regulatory network.

4.1 Benchmark example

For the numerical illustrative example, we consider \(i = 1, m = 1\) and the following stochastic differential equation of the second order:
\[
dx_t = [Ax_t + Bu_t]dt + (Cx_t + Du_t + \sigma)dW_t
\]
with $A, C \in \mathbb{R}^{2 \times 2}$, $B, D, \sigma \in \mathbb{R}^{2 \times 1}$ and

$$u(t) = Kx_t, \quad K \in \mathbb{R}^{1 \times 2},$$

so that

$$dx_t = [A + BK]x_t dt + [(C + DK)x_t + \sigma] dW_t.$$  \hfill (4.1)

By (4.1), for the matrices (3.3), (3.4) we have

$$A = \begin{bmatrix} 0.4 & -0.8 \\ 1.8 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1.9 & -1.8 \\ 0.8 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}.$$

We also have

$$\hat{W} := \begin{pmatrix} \tilde{W}_2(3) & C^T P + \theta_1^T \\ PC + \theta_1 & P - \kappa I_{2 \times 2} \end{pmatrix},$$

$$\tilde{W}_2(3) := \begin{bmatrix} P \left( A + \frac{\alpha}{2} \left( 1 + \frac{\tau}{\beta} \right) I_{n \times n} \right) + \left( A + \frac{\alpha}{2} \left( 1 + \frac{\tau}{\beta} \right) I_{n \times n} \right)^T P \right] + \theta_2 + \theta_2^T + Q_1,$$ \hfill (4.2)

where $Q_1 \succeq 0$ satisfies the additional inequality (3.3)

$$\begin{bmatrix} Q_1 & C^T P + \theta_1^T \\ PC + \theta_1 & P \end{bmatrix} \succeq 0,$$

$$\theta_1 = PDK, \quad \theta_2 = PBK.$$

Consider now three case studies:

- the first one is free of any restrictions ($\tau = 0$);
- in the second one, we try to obtain a smaller ellipsoidal region using the parameter $\tau$ as an additional tool for the optimization process;
- in the last case, we show the state convergence when $\beta = 0$, i.e. when uncertainties are absent, and we guarantee the convergence to the origin with probability one.

When $\tau = 0$, in (4.2),

$$\tilde{W}_2(3) := \begin{bmatrix} P \left( A + \frac{\alpha}{2} I_{n \times n} \right) + \left( A + \frac{\alpha}{2} I_{n \times n} \right)^T P \right] + \theta_2 + \theta_2^T + Q_1.$$

The numerical implementation of the noisy signal in Simulink is given in terms of a Gaussian noise signal generator $W_t$, using the following approximation:

$$\frac{\Delta W_t}{\Delta t} \approx \frac{W_t - W_{t-h}}{h},$$

where $0 < h \ll 1$. Figures 2 and 3 show the state trajectories within the corresponding ellipsoid.
FIG. 2. The trajectories of state.

FIG. 3. The trajectory of state in the corresponding ellipsoid.
The numerical solutions of the optimization problem (3.1) are

\[
P^* = \begin{bmatrix} 0.2384 & -0.0000 \\ -0.0000 & 0.2384 \end{bmatrix}, \quad K^* = [-5.2662 \quad -2.6492],
\]

\[
\beta^+ = 0.2622, \quad \alpha^* = 0.5233, \quad \kappa^* = 0.26.
\]

When the restricting parameter \( \tau \) is positive, we obtain the above problem formulation as in (4.2) and (3.3). The corresponding state trajectories are depicted in Figs 4 and 5.

In this case,

\[
P^* = \begin{bmatrix} 0.0259 & -0.0000 \\ -0.0000 & 0.0259 \end{bmatrix}, \quad K^+ = [-434.3058 \quad -268.8806],
\]

\[
\tau^* = 0.001, \quad \beta^* = 0.0285, \quad \alpha^* = 13.4588, \quad \kappa^* = 0.1.
\]

The last case, \( \beta = 0 \), is illustrated in Fig. 6

\[
K^* = [2.9832 \quad -20.8990], \quad \alpha^* = 98.
\]

4.2 A gene regulation system

Consider a model of a gene regulatory network as in Example 2.1, which is affected by the presence of stochastic noise. Using the approach presented in El Samad & Khammash (2004), our aim is actually to stabilize (2.2) to the equilibrium point

\[
(R^*, P^*_o) = \left( \frac{K_1 \gamma P_o}{K_2 K_P + \gamma \gamma P_o}, \frac{K_1 K_{P_0}}{K_2 K_P + \gamma \gamma P_o} \right),
\]

\[\text{Fig. 4. The trajectory of state in the corresponding ellipsoid, in the case } \tau \neq 0.\]
FIG. 5. The trajectory of state in the corresponding ellipsoid, in the case $\tau \neq 0$.

FIG. 6. The trajectory of state in probability one.
with the coordinate change given by

\[ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} R - R^* \\ P_o - P_o^* \end{bmatrix}, \]

where \( R \) and \( P_o \) are the actual system states. As a consequence, if we wish to stabilize \( x^T(t) := [x_1, x_2]^T \) to the origin, it means that we wish to show that the state \( x(t) \) will converge to the equilibrium point \( (R^*, P_o^*) \). In terms of the new variables, we have

\[ \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -\gamma R & -K_2 \\ K_{P_o} & -\gamma P_o \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt. \quad (4.3) \]

When the parameters \(-\gamma R, -K_2, K_{P_o}\) and \(-\gamma P_o\) are perturbed by a stochastic Gaussian noise \( \{W_t\}_{t \geq 0}\), we can finally rewrite (4.3) as

\[ \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -\gamma R & -K_2 \\ K_{P_o} & -\gamma P_o \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dW_t. \]

On the other hand, if we consider the additive disturbances (such as measurement noise or environmental variations), the above equation becomes

\[ \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -\gamma R & -K_2 \\ K_{P_o} & -\gamma P_o \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} dW_t. \]

Taking \( u(t) = \tilde{K} x(t) \) implies that

\[ \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -\gamma R & -K_2 \\ K_{P_o} & -\gamma P_o \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dt + \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} dW_t. \quad (4.4) \]

Denote

\[ A = \begin{bmatrix} -\gamma R & -K_2 \\ K_{P_o} & -\gamma P_o \end{bmatrix}, \quad C = \begin{bmatrix} -\delta_1 & -\delta_2 \\ -\delta_3 & -\delta_4 \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \]

and for (4.4), consider the following matrix values:

\[ A = \begin{bmatrix} -0.35 & 0.001 \\ 0.8 & -0.0049 \end{bmatrix}, \quad C = \begin{bmatrix} -0.8 & -0.1 \\ 0.2 & -0.5 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}. \]

Following the same procedure as in the first numerical example (see (4.2)), we obtain the following results:

\[ P_o^* = \begin{bmatrix} 0.0115 & -0.0000 \\ -0.0000 & 0.0115 \end{bmatrix}, \quad \tilde{K}^* = \begin{bmatrix} -24.2925 & 0 \\ 0 & -24.6376 \end{bmatrix}, \quad \tau^* = 0.01, \quad \beta^* = 0.0127, \quad \alpha^* = 0.0152, \quad \kappa = 0.28. \]

The corresponding trajectories of \( x(t) \) are presented below in Figs 7–9.
FIG. 7. The trajectory of state $x_1$.

FIG. 8. The trajectory of state $x_2$. 
5. Conclusions

This paper proposes the implemented algorithm for the designing of a robust stabilizing control for the class of LCSDEs with uncertainties. As a result of the suggested numerical optimization procedure, we construct an attractive ellipsoid of a ‘minimal size’ and calculate numerically the corresponding ‘optimal’ gain matrix. The computational algorithm proposed in this work makes it possible to generate the ‘best’ proportional feedback control law. Two examples illustrate the effectiveness of the suggested approach.

REFERENCES


