Chapter 1
The fault detection problem in nonlinear systems using residual generators

Abstract In this chapter we study the fault detection problem using residual generators based upon high gain nonlinear observers in a differential algebraic framework. We analyze the stability of the residual generator when a fault occurs. We also consider two faults types: constant and time-varying faults. It is shown that under some mild conditions over the aforementioned faults the residual is different from zero.

1.1 Introduction

The high reliability required in industrial processes has created the necessity of detecting abnormal conditions while processes are operating. These conditions are called faults and it is important to detect and to isolate them in the early stages. The fault is a term which means degradation of the process or degradation in equipment performance because of changes in the process physical characteristics, process inputs or environmental conditions. A fault in a process is considered as a not-allowable deviation which can be detected by an appropriated signal evaluation. State observers are suitable structures to evaluate this change. The difference between the measured outputs of the process and the observer is the so-called residual value which is used to detect the fault. In this chapter we consider the fault detection problem with a residual generator approach using high gain nonlinear observers in a differential algebraic framework. We study two types of faults: constant and time-varying faults. The differential algebraic approach allows us to define the concept of algebraic observability [3, 4, 5, 6] and supplies state estimation through observers designed for systems described by differential algebraic equations [5, 11, 13, 14, 15, 16, 17, 18]. The chapter is organized as follows. Section 1.2 presents some differential algebra basic definitions. In Section 1.3, we present the residual generation problem and the residual generator stability using the uniform ultimate boundedness (UUB) theorem [2]. Section 1.4 presents two fault cases. A numerical example is given in Section 1.5.
1.2 Observation problem

Consider the following nonlinear system in the so-called generalized observability canonical form (GOCF) [7, 12, 13]:

\[
\def\sym#1#2{\mathcal{#1}_{#2}}
\begin{align*}
\dot{\xi}_i &= \xi_{i+1}, & 1 \leq i \leq n-1 \\
\dot{\xi}_n &= -L_0(\xi, u, \ldots, u^\nu) \\
y &= \xi_1
\end{align*}
\tag{1.1}
\]

where \( L_0 \) is a \( C^1 \) real-valued function, \( \xi = \col(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y(t) \in \mathbb{R} \) and some integer \( \nu \geq 0 \).

Remark 1.1. In general, a nonlinear system

\[
\begin{align*}
\dot{x}(t) &= g(x, u) \\
y(t) &= h(x, u)
\end{align*}
\tag{1.2}
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R} \), \( g(\cdot, \cdot) \) and \( h(\cdot, \cdot) \) are polynomial functions of their arguments, may be transformed to the GOCF described by (1.1) as a consequence of the differential primitive element for nonlinear systems [3, 7, 12, 13, 19].

System (1.1) may be written in compact form as

\[
\begin{align*}
\dot{\xi}(t) &= A\xi + \phi(\xi, \bar{u}) \\
y(t) &= C\xi
\end{align*}
\tag{1.3}
\]

where \( C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \), \( \bar{u} = \begin{pmatrix} u & u^{(1)} & \ldots & u^{(\nu)} \end{pmatrix} \). The elements of \( A \) are given by

\[
A_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
\phi(\xi, \bar{u}) = \col(0 \ldots 0 - L_0(\xi, \bar{u}))
\tag{1.4}
\]

is continuously differentiable. Hence, an estimate \( \hat{\xi} \) of \( \xi \) can be given by an exponential nonlinear observer (\( \mathcal{O} \)) of the form

\[
(\mathcal{O}) : \quad \dot{\hat{\xi}} = A\hat{\xi} + \phi(\hat{\xi}, \bar{u}) - S_\theta^{-1}C^T C(\hat{\xi} - \xi)
\tag{1.5}
\]

where

\[
\phi(\hat{\xi}, \bar{u}) = \col(0 \ldots 0 - L_0(\hat{\xi}, \bar{u}))
\tag{1.6}
\]

and \( S_\theta \) is the positive definite solution of [8]

\[
S_\theta \left( A + \frac{\theta}{2} I \right) + \left( A^T + \frac{\theta}{2} I \right) S_\theta = C^T C
\tag{1.7}
\]

for some \( \theta > 0 \). The coefficients of \( (S_\theta)_{ij} \) are given by
where \((\alpha_{ij})\) are the entries of a symmetric positive-definite matrix which does not depend on \(\theta\), and \(C = (1 \ 0 \ldots \ 0)\).

Now, from (1.3) and (1.5), the estimation error dynamic \(\epsilon = \xi - \hat{\xi}\) is given by

\[
\dot{\epsilon} = (A - S_\theta^{-1}C^T C)\epsilon + \Phi(\epsilon, \bar{u}) \tag{1.9}
\]

where

\[
\Phi(\epsilon, \bar{u}) = \phi(\hat{\xi} + \epsilon, \bar{u}) - \phi(\hat{\xi}, \bar{u}) \tag{1.10}
\]

Now, first of all, we introduce the following notation and definitions. Denote

\[
\|x\|_{S_\theta} = (x^T S_\theta x)^{1/2}, S_\theta \text{ being the solution of (1.7). Then, if } \Phi(\epsilon, \bar{u}) \text{ is differentiable, we get } \|\Phi(\epsilon, \bar{u})\|_{S_\theta} \leq \gamma \|\epsilon\|_{S_\theta}, \text{ for some } \gamma > 0. \]

In what follows, we present a result which shows some characteristics and structural properties of matrix \((A - S_\theta^{-1}C^T C)\).

**Lemma 1.1.** \(A_\theta = (A - S_\theta^{-1}C^T C)\) is a Hurwitz matrix. Furthermore, the characteristic polynomial of \(A_\theta\) is \(P(\lambda) = (\lambda + \theta)^n\).

**Proof.** We start with a simple case, that is to say, a single-input single-output system with two states \((S_\theta)_{2\times2}\). Matrix \(S_\theta\) is given by

\[
S_\theta = \begin{pmatrix}
1 & -1/	heta^2 \\
-1 & 2/	heta^2
\end{pmatrix}
\]

thus,

\[
S_\theta^{-1} = \begin{pmatrix}
2\theta & \theta^2 \\
\theta^2 & \theta^3
\end{pmatrix}
\]

and

\[
S_\theta^{-1}C^T C = \begin{pmatrix}
2\theta & 0 \\
\theta^2 & 0
\end{pmatrix}
\]

Then, matrix \(A_\theta\) is given as

\[
A_\theta = \begin{pmatrix}
-2\theta & 1 \\
-\theta^2 & 0
\end{pmatrix} \tag{1.11}
\]

Then its eigenvalues are given by \(\lambda_1 = \lambda_2 = -\theta\). For the case \((S_\theta)_{3\times3}\) we have

\[
S_\theta = \begin{pmatrix}
1 & -1/	heta^2 & 1/	heta^3 \\
-1 & 2/	heta^2 & 3/	heta^4 \\
1 & -3/	heta^3 & 6/	heta^5
\end{pmatrix} \tag{1.12}
\]

with inverse matrix
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\[ S_{\theta}^{-1} = \begin{pmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{pmatrix}, \]

\[ S_{\theta}^{-1}C^TC = \begin{pmatrix} 3\theta & 0 & 0 \\ 3\theta^2 & 0 & 0 \\ \theta^3 & 0 & 0 \end{pmatrix} \]

and matrix \( A_{\theta} \) is given by

\[ A_{\theta} = \begin{pmatrix} -3\theta & 1 & 0 \\ -3\theta^2 & 0 & 1 \\ -\theta^3 & 0 & 0 \end{pmatrix} \] (1.13)

with eigenvalues \( \lambda_1 = \lambda_2 = \lambda_3 = -\theta \).

Finally, by induction, the matrix \( A_{\theta} \) for the case \( n \times n \) is given by

\[
A_{\theta} = \begin{pmatrix}
-n\theta & 1 & 0 & \cdots & 0 & 0 & 0 \\
-n(n-1)/\theta^2 & 0 & 1 & \cdots & 0 & 0 & 0 \\
-n(n-1)(n-2)/3! & 0 & 0 & \cdots & 0 & 0 & 0 \\
& \vdots & & \ddots & & \ddots & \ddots \\
-n(n-1)(n-2)\cdots(n-(n-(r-1)))/(n-(r-1))! & \theta^{n-(n-(r-1))} & 0 & \cdots & 0 & 0 & 0 \\
& (n-(r-1))!(r-1)! & \theta^{n-(n-(r-1))} & 0 & \cdots & 0 & 0 & 0 \\
& n(n-1)(n-2)\cdots(n-(n-(r+1)))/(n-(r+1))! & \theta^{n-(n-(r+1))} & 0 & \cdots & 0 & 0 & 0 \\
& & \vdots & & \ddots & & \ddots & \ddots \\
& -n(n-1)(n-2)\cdots(n-(r-1))/2! & \theta^{n-(r-1)} & 0 & \cdots & 0 & 1 & 0 \\
& -n(n-1)/\theta^{n-2} & 0 & \cdots & 0 & 1 & 0 & 0 \\
& -\theta^n & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (1.14)

which has the following eigenvalues:

\[ \lambda_1 = \lambda_2 = \cdots = \lambda_{r-1} = \lambda_r = \lambda_{r+1} = \cdots = \lambda_{n-1} = \lambda_n = -\theta \]

The above means that \( A_{\theta} \) is a Hurwitz matrix and the characteristic polynomial is given by

\[ P(\lambda) = \det(\lambda I - A_{\theta}) = (\lambda + \theta)^n \]
1.3 Fundamental problem of residual generation

The fault detection scheme is composed of a residual generator and a fault mode rule. The residual generator is a filter whose inputs correspond to the inputs and outputs of the plant. The filter outputs signals called residual values are used for fault detection purposes in the following way: if the residual values are zero, the process has no fault, otherwise the process presents a fault.

Now, we consider a nonlinear system from (1.2) with an additional fault

\[ \dot{x}(t) = g(x,u) + v(t) \]
\[ y(t) = h(x,u) \]  

(1.15)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) = (u_1(t), \ldots, u_m(t))^T \in \mathcal{U} \subset \mathbb{R}^m \) is the input vector, \( v(t) \) is a scalar function representing a system fault. For the sake of simplicity we consider that only one fault occurs at a given time, \( y(t) \in \mathbb{R} \) is the measured output vector. \( g(x,u) \), and \( h(x,u) \) are functions of class \( C^\infty \). \( u(t) \) and \( y(t) \) are the system input and output signals, respectively, and we suppose that they are known. Function \( v(t) \) is unknown and arbitrary and belongs to a compact set.

In this work, we define a good input to a system input such that the coordinate transformation is not singular and it can carry out the system (1.15) into the GOCF (1.3).

A residual generator can be defined [9, 10] as a nonlinear dynamic system given by

\[ \dot{z}(t) = G(z,y,u) \]
\[ r(t) = H(y,z) \]  

(1.16)

where \( z(t) \in \mathbb{R}^\bar{n} \) is the state vector, \( r(t) \in \mathbb{R}^\bar{p} \) is the output vector, \( u \) and \( y \) are the inputs to the system and corresponding to input and output vectors of (1.15). A residual generator must satisfy the following conditions:

(C1) If \( v(t) = 0 \) for each initial condition \((x(0),z(0))\) of the extended system (1.15)-(1.16) and for all admissible good input \( u \), \( \lim_{t \to \infty} r = 0 \). Then, in the absence of fault, \( r \) asymptotically converges to zero.

(C2) If \( v(t) \neq 0 \) for all \( t \geq t_0 \) then \( r(t) \neq 0 \) for all \( t \geq t_0 \).

If the above conditions are satisfied, we then say that \( r \) is a residual value.

1.3.1 Residual generator stability

In this part, we consider the stability analysis of the residual generator for the system with fault (1.15) using the uniform ultimate stability (or UUB, uniform ultimate boundedness) theorem, given in [2].
Lemma 1.2. Suppose that there exists a differential primitive element such that it is possible to transform system (1.15) into the GOCF given by
\[
\dot{\xi}(t) = A\xi + \varphi(\xi, \bar{u}) + W(t) \\
y(t) = C\xi
\]
where the term \( W(t) = \text{col} \left[ 0 \ldots 0 w(t) \right] \) is the fault in the transformed system and it is supposed to be bounded. Furthermore, the following system:
\[
\dot{\hat{\xi}}(t) = A\hat{\xi} + \varphi(\hat{\xi}, \bar{u}) + S^{-1}_\theta \Phi(\xi, \bar{u}) + S^{-1}_\theta CT(y - \hat{y}) \\
\hat{y}(t) = C\hat{\xi}
\]
(1.18)
is an observer for (1.17), with \( S_\theta \) the gain matrix satisfying (1.7) and \( r \) is the residual value. Then, the estimation error dynamics given by
\[
\dot{\epsilon} = A_\theta \epsilon + \Phi(\epsilon, \bar{u}) + W(t) \\
r = C\epsilon
\]
(1.19)
is UUB, where \( \Phi(\epsilon, \bar{u}) \) is given in (1.10), \( A_\theta = (A - S^{-1}_\theta CT C) \) and \( \epsilon \) belongs to the compact set \( B_b = \{ \epsilon \mid \|\epsilon\| \leq b, b > 0 \} \).

Proof. Consider the following Lyapunov function candidate for system (1.19):
\[
V(t) = \epsilon^T S_\theta \epsilon > 0
\]
Taking the time derivative we have
\[
\dot{V}(t) = \dot{\epsilon}^T S_\theta \epsilon + \epsilon^T S_\theta \dot{\epsilon} \\
= \left[ \epsilon^T (A^T - C^T S^{-1}_\theta C) \epsilon + \Phi^T(\epsilon, \bar{u}) + W^T \right] S_\theta \epsilon \\
+ \epsilon^T S_\theta \left[ (A - S^{-1}_\theta CT C) \epsilon + \Phi(\epsilon, \bar{u}) + W \right] \\
= \epsilon^T A^T S_\theta \epsilon - \epsilon^T C^T C \epsilon + \Phi^T(\epsilon, \bar{u})S_\theta \epsilon \\
+ W^T S_\theta \epsilon + \epsilon^T S_\theta (A - S^{-1}_\theta CT C) \epsilon + \epsilon^T S_\theta \Phi(\epsilon, \bar{u}) + \epsilon^T S_\theta W \\
= \epsilon^T (A^T S_\theta + S_\theta A - C^T C) \epsilon - \epsilon^T C^T C \epsilon \\
+ \Phi^T(\epsilon, \bar{u})S_\theta \epsilon + \epsilon^T S_\theta \Phi(\epsilon, \bar{u}) + W^T S_\theta \epsilon + \epsilon^T S_\theta W
\]
For the sake of simplicity we have dropped the argument \( t \) in \( W(t) \). Since
\[
A^T S_\theta + S_\theta A - C^T C = -\theta S_\theta
\]
(1.20)
we have
\[
\dot{V}(t) = -\theta \epsilon^T S_\theta \epsilon - \epsilon^T C^T C \epsilon + 2\epsilon^T S_\theta \Phi(\epsilon, \bar{u}) + 2\epsilon^T S_\theta W
\]
using the fact \( \epsilon^T C^T C \epsilon > 0 \), \( \dot{V}(t) \) is upper bounded as
\[ V(t) \leq -\theta \epsilon^T S_{\theta} \epsilon + 2\epsilon^T S_{\theta} \Phi(\epsilon, \bar{u}) + 2\epsilon^T S_{\theta} W \]  

(1.21)

First, we will consider each term in (1.21). The term \(-\theta \epsilon^T S_{\theta} \epsilon\) corresponds to

\[ -\theta \epsilon^T S_{\theta} \epsilon = -\theta \| \epsilon \|_{S_{\theta}} \]  

(1.22)

Using the Cholesky decomposition for a symmetric positive-definite matrix \(S_{\theta}\), the norm for the second term is

\[ \| \epsilon^T S_{\theta} \Phi(\epsilon, \bar{u}) \| = \| \epsilon^T MMT \Phi(\epsilon, \bar{u}) \| \]

with \(S_{\theta} = MMT\). Now making \(\tilde{\epsilon}^T = \epsilon^T M\), we have \(\tilde{\epsilon}^T = M^T \epsilon\). Moreover, with \(\Phi(\epsilon, \bar{u}) = M^T \Phi(\epsilon, \bar{u})\) and \(\tilde{\Phi}(\epsilon, \bar{u}) = \Phi^T(\epsilon, \bar{u})M\), it can be seen that

\[ \| \tilde{\epsilon} \| = \left( \tilde{\epsilon}^T \tilde{\epsilon} \right)^{1/2} = \left( \epsilon^T MMT \epsilon \right)^{1/2} \]
\[ = \left( \epsilon^T S_{\theta} \epsilon \right)^{1/2} = \| \epsilon \|_{S_{\theta}} \]

\[ \| \Phi(\epsilon, \bar{u}) \| = \left( \Phi^T(\epsilon, \bar{u}) \Phi(\epsilon, \bar{u}) \right)^{1/2} \]
\[ = \left( \Phi^T(\epsilon, \bar{u}) MMT \Phi(\epsilon, \bar{u}) \right)^{1/2} \]
\[ = \left( \Phi^T(\epsilon, \bar{u}) S_{\theta} \Phi(\epsilon, \bar{u}) \right)^{1/2} \]
\[ = \| \Phi(\epsilon, \bar{u}) \|_{S_{\theta}} \]

and

\[ \| \epsilon^T S_{\theta} \Phi(\epsilon, \bar{u}) \| = \| \epsilon^T MMT \Phi(\epsilon, \bar{u}) \| \]
\[ = \| \tilde{\epsilon}^T \Phi(\epsilon, \bar{u}) \| \]

Then, using the Cauchy-Schwarz inequality we have

\[ \| \epsilon^T S_{\theta} \Phi(\epsilon, \bar{u}) \| \leq \| \tilde{\epsilon} \| \| \Phi(\epsilon, \bar{u}) \| \]
\[ \leq \| \tilde{\epsilon} \| \| \Phi(\epsilon, \bar{u}) \|_{S_{\theta}} \]

Since \(\Phi(\epsilon, \bar{u})\) is differentiable we get

\[ \| \Phi(\epsilon, \bar{u}) \|_{S_{\theta}} \leq \gamma \| \epsilon \|_{S_{\theta}} \]

and

\[ \| \epsilon^T S_{\theta} \Phi(\epsilon, \bar{u}) \| \leq \gamma \| \epsilon \|^2_{S_{\theta}} \]  

(1.23)

Finally, for the third term in (1.21), a bound is obtained as follows:

\[ \| \epsilon^T S_{\theta} W \| = \| \epsilon^T MMT W \| = \| \epsilon^T M^T W \| \]
Using the Cauchy-Schwarz inequality we obtain
\[ \| e^T S_\theta W \| = \| \bar{e}^T M^T W \| \leq \| \bar{e} \| \| M^T W \| \]
Defining \( \rho = W \) and making \( \tilde{\rho} = M^T W = M^T \rho, \tilde{\rho}^T = \rho^T M \), we have
\[ \| \tilde{\rho} \| = (\tilde{\rho}^T \tilde{\rho})^{1/2} = (\rho^T M M^T \rho)^{1/2} = (\rho^T S_\theta \rho)^{1/2} = \| \rho \|_{S_\theta} \]
That is to say,
\[ \| M^T W \| = \| W \|_{S_\theta} \]
then, the third term (1.21) is bounded as
\[ \| e^T S_\theta W \| \leq \| \bar{e} \| \| M^T W \| \leq \| e \|_{S_\theta} \| W \|_{S_\theta} \]
Now, we assume that the fault is bounded, i.e.
\[ \| W \|_{S_\theta} \leq \Gamma \]
where \( \Gamma > 0 \), then we get
\[ \| e^T S_\theta W \| \leq \Gamma \| e \|_{S_\theta} \tag{1.24} \]
Then, using (1.22)-(1.24), it readily follows that the time derivative \( \dot{V}(t) \) remains bounded, that is to say
\[ \dot{V}(t) \leq -\theta \| e \|_{S_\theta}^2 + 2\gamma \| e \|_{S_\theta}^2 + 2\Gamma \| e \|_{S_\theta} \tag{1.25} \]
It is clear that if the fault is zero, i.e. \( W = 0 \), this leads us to obtain the particular case of exponential convergence of the high gain nonlinear observer,
\[ \dot{V}(t) \leq -\theta \| e \|_{S_\theta}^2 + 2\gamma \| e \|_{S_\theta}^2 = -(\theta - 2\gamma)V(t) \]
i.e.
\[ \| e \|_{S_\theta} \leq \| e(0) \|_{S_\theta} e^{-\left(\frac{\theta}{2} - \gamma\right)t} \tag{1.26} \]
with \( \theta > 2\gamma \). Now, if \( W \neq 0 \), consider inequality (1.25) then
\[ \dot{V}(t) \leq -(\theta - 2\gamma)\| e \|_{S_\theta}^2 + 2\Gamma \| e \|_{S_\theta} \]
Using the Rayleigh-Ritz inequality
\[ \lambda_{\min}(S_\theta)\| e \|^2 \leq \| e \|_{S_\theta}^2 \leq \lambda_{\max}(S_\theta)\| e \|^2 \]
we conclude that \( \dot{V}(t) \) satisfies
\[ \dot{V}(t) \leq -(\theta - 2\gamma)\lambda_{\min}(S_\theta)\| e \|^2 + 2\Gamma \sqrt{\lambda_{\max}(S_\theta)}\| e \| \]
By applying the UUB theorem [2], it directly follows that $\epsilon(t)$ is bounded uniformly for any initial state $\epsilon(0)$, and $\epsilon(t)$ remains in a compact set $B_b = \{\epsilon \mid \|\epsilon\| \leq b, b > 0\}$ where

$$b = \sqrt{\frac{\lambda_{\text{max}}(S_\theta)}{\lambda_{\text{min}}(S_\theta)}} \left(\frac{2\Gamma}{(\theta - 2\gamma)\lambda_{\text{min}}(S_\theta)}\right) > 0$$

\[\square\]

### 1.4 Fault detection

In the next subsections we will study fault cases: namely, constant fault and time-varying fault.

#### 1.4.1 Constant fault case

We consider system (1.15) in the GOCF (1.17). From Lemma 1.2, an observer for this system is given by (1.18) with estimation error dynamics (1.19). First, we tackle the constant fault case problem with a lemma related with the existence and uniqueness of solutions of the error dynamics (1.17).

**Lemma 1.3.** Consider the estimation error dynamics (1.19)

$$\dot{\epsilon} = A_\theta \epsilon + \Phi(\epsilon, \bar{u}) + W(t)$$

$$r = C\epsilon$$

(1.27)

when the fault $W(t)$ is constant. i.e,

$$W(t) = W_c = \text{col} \left(0 \ 0 \ldots w_c\right), \quad |w_c| > 0$$

Then, there exists a unique constant solution $\epsilon_s$ for all $t \in [0, \infty)$.

**Proof.** The proof is split into two parts.

(a) **Existence.** Since $\Phi(\epsilon, \bar{u})$ is differentiable, then there exists a solution for (1.27) for all $t \in [0, \infty)$ [1]

(b) **Uniqueness.** First, let $\epsilon_s$ be a solution of the algebraic equation

$$A_\theta \epsilon_s + \Phi(\epsilon_s, \bar{u}) + W_c = 0$$

(1.28)

with
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\[ \Phi(\varepsilon_s, \bar{u}) = \phi(\hat{\xi} + \varepsilon_s, \bar{u}) - \phi(\hat{\xi}, \bar{u}) \]

\[ = \text{col} \left( 0 \ldots 0 -L_0(\hat{\xi} + \varepsilon_s) + L_0(\hat{\xi}) \right) \]

\[ = \text{col} \left( 0 \ldots 0 -L_0(\hat{\xi} + \varepsilon_s) + L_0(\hat{\xi}) \right) \]

and define \( z = \varepsilon - \varepsilon_s \), then, using (1.27) and (1.28), we have

\[ \dot{z} = \dot{\varepsilon} - \dot{\varepsilon}_s \]

\[ = A_\theta \varepsilon(t) + \Phi(\varepsilon, \bar{u}) + W_c \]

\[ = A_\theta(z + \varepsilon_s) + \Phi(z + \varepsilon_s, \bar{u}) + W_c \]

From (1.28), we replace \( A_\theta \varepsilon_s \),

\[ \dot{z} = A_\theta z + \Phi(z + \varepsilon_s, \bar{u}) - \Phi(\varepsilon_s, \bar{u}) \quad (1.29) \]

Consider the Lyapunov function candidate \( V = z^T S_\theta z \). Then,

\[ \dot{V} = z^T S_\theta \dot{z} + z^T S_\theta \dot{z} \]

\[ = [z^T A^T_\theta + \Phi^T(z + \varepsilon_s, \bar{u}) - \Phi^T(\varepsilon_s, \bar{u})] S_\theta z \]

\[ + z^T S_\theta [A_\theta z + \Phi(z + \varepsilon_s, \bar{u}) - \Phi(\varepsilon_s, \bar{u})] \]

Substituting \( A_\theta = (A - S_\theta^{-1} C^T C) \), we have

\[ \dot{V} = z^T (A^T S_\theta + S_\theta A - C^T C) z - z^T C^T C z \]

\[ + 2[\Phi(z + \varepsilon_s, \bar{u}) - \Phi(\varepsilon_s, \bar{u})] S_\theta z \]

Substituting (1.20) we obtain

\[ \dot{V} = -\theta z^T S_\theta z - z^T C^T C z \]

\[ + 2[\Phi(z + \varepsilon_s, \bar{u}) - \Phi(\varepsilon_s, \bar{u})] S_\theta z \]

Since \( z^T C^T C z > 0 \), \( z^T S_\theta z = ||z||^2_{S_\theta} \), as well as \( \Phi(\varepsilon_s, \bar{u}) \) is differentiable, then

\[ \dot{V} \leq -\theta ||z||^2_{S_\theta} + 2\gamma ||z||^2_{S_\theta} \]

\[ = -\left( \theta - 2\gamma \right) ||z||^2_{S_\theta} \]

from which it follows that \( z \) converges exponentially to zero for \( \theta > 2\gamma \); then \( \varepsilon \) converges to \( \varepsilon_s \). □

Using results from Lemma 1.1 we establish the following theorem.

Theorem 1.1. Consider (1.28):
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\[ A_\theta \varepsilon_s + \Phi(\varepsilon_s, \bar{u}) + W_c = 0 \]  

(1.30)

with solution \( \varepsilon_s, \|W_c\| > 0 \), and \( \Phi(\varepsilon, \bar{u}) \) is differentiable, i.e.,

\[ \|\Phi(\varepsilon, \bar{u})\| \leq \gamma \|\varepsilon\|. \]  

(1.31)

Then, the residual value \( r_s = C\varepsilon_s \) satisfies the following inequality:

\[ |\omega_c| \leq \left( \theta^n + \gamma \sqrt{H(\theta)} \right) |r_s| \]  

(1.32)

where \( \|\varepsilon_s\| = \sqrt{H(\theta)} |r_s| \) with \( H(\theta) \) a positive function for \( \theta > 0 \).

Proof. From (1.30) and (1.14) we have

\[
\begin{pmatrix}
-n\theta & 1 & 0 & \cdots & 0 & 0 \\
-n(n-1)\theta^2 & 0 & 1 & \cdots & 0 & 0 \\
-n(n-1)(n-2)\theta^3 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-n(n-1)(n-2)\cdots(n-(n-(r-1)))!\theta^{n-(n-(r-1))} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-n(n-1)(n-2)\cdots(n-(n-(r+1)))!\theta^{n-(n-(r+1))} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-n(n-1)\theta^{n-2} & 0 & 0 & \cdots & 0 & 1 \\
-n\theta^{n-1} & 0 & 0 & \cdots & 0 & 0 \\
-\theta^n & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\vdots \\
\varepsilon_{n(n-r)} \\
\vdots \\
\varepsilon_{n-1} \\
\varepsilon_n \\
\end{pmatrix}
+ \begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\vdots \\
0 \\
0 \\
\end{pmatrix}

= 0
\]

(1.33)
The fault detection problem in nonlinear systems using residual generators

\[ \epsilon_{s_2} = \epsilon_{s_1} (n\theta) \]
\[ \epsilon_{s_3} = \epsilon_{s_1} \left( \frac{n(n-1)}{2!} \theta^2 \right) \]
\[ \vdots \]
\[ \epsilon_{s_{n-(n-r)}} = \epsilon_{s_1} \left( \frac{n(n-1)(n-2) \ldots (n-(n-r))!}{(n-r)!r!} \theta^{n-(n-r)} \right) \]
\[ \vdots \]
\[ \epsilon_{s_{n-1}} = \epsilon_{s_1} \left( \frac{n(n-1)}{2!} \theta^{n-2} \right) \]
\[ \epsilon_{s_n} = \epsilon_{s_1} (n\theta^{n-1}) \]

\[ \omega - \tilde{L}_0(\epsilon_s, \tilde{u}) = \epsilon_{s_1} \theta^n \]

where \( \epsilon_s = (\epsilon_{s_1} \epsilon_{s_2} \ldots \epsilon_{s_{n-1}} \epsilon_{s_n})^T \). Then, the norm of \( \epsilon_s \) is given by

\[ \|\epsilon_s\| = \epsilon_{s_1}^2 + \epsilon_{s_2}^2 + \epsilon_{s_3}^2 + \ldots + \epsilon_{s_{n-1}}^2 + \epsilon_{s_n}^2 \]

\[ = \epsilon_{s_1}^2 + \epsilon_{s_1}^2 (n\theta)^2 + \epsilon_{s_1}^2 \left( \frac{n(n-1)}{2!} \theta^2 \right)^2 + \ldots \]

\[ + \epsilon_{s_1}^2 \left( -\frac{n(n-1)(n-2) \ldots (n-(n-r))!}{(n-r)!r!} \theta^{n-(n-r)} \right)^2 + \ldots \]

\[ + \epsilon_{s_1}^2 \left( \frac{n(n-1)}{2!} \theta^{n-2} \right)^2 + \epsilon_{s_1}^2 \left( \frac{n(n-1)}{2!} \theta^{n-2} \right)^2 \]

\[ = \epsilon_{s_1}^2 \left[ 1 + (n\theta)^2 + \left( \frac{n(n-1)}{2!} \theta^2 \right)^2 + \ldots \right. \]

\[ + \left( \frac{n(n-1)(n-2) \ldots (n-(n-r))!}{(n-r)!r!} \theta^{n-(n-r)} \right)^2 + \ldots \]

\[ + \left( \frac{n(n-1)}{2!} \theta^{n-2} \right)^2 + (n\theta^{n-1})^2 \right] \]

which is written compactly as

\[ \|\epsilon_s\| = H(\theta) \epsilon_{s_1}^2 \]

or

\[ \|\epsilon_s\| = \sqrt{H(\theta)} |r_s| \]

(1.35)

where \( r_s = C\epsilon_s = \epsilon_{s_1} \) and
\[ H(\theta) = \left[ 1 + (n\theta)^2 + \left( \frac{n(n-1)}{2!} \theta^2 \right)^2 + \ldots + \left( \frac{n(n-1)(n-2)\ldots(n-(n-r))!}{(n-r)!r!} \theta^{n-(n-r)} \right)^2 + \ldots \right. \\
\left. + \left( \frac{n(n-1)^2}{2!} \right)^2 + (n\theta^{n-1})^2 \right] \]

Note that \( H(\theta) > 0 \) for \( \theta > 0 \). Now, from (1.34), we have

\[ \omega_c - \bar{L}_0(\varepsilon_s, \bar{u}) = \varepsilon_{s1} \theta^n \]

Note that \( \|\bar{L}(\varepsilon_s, \bar{u})\| = \|\Phi(\varepsilon_s, \bar{u})\| \leq \gamma \|\varepsilon_s\| \). Then

\[ |\omega_c| \leq \left( \theta^n + \gamma \sqrt{H(\theta)} \right) |r_s| \]

\[ \square \]

1.4.2 Time-varying fault case

In this case, the fault term is considered as \( W(t) = W_c + F(t) \) where \( W_c = \text{col} \left( 0 0 \ldots \omega_c \right) \) and \( F(t) = \text{col} \left( 0 0 \ldots f(t) \right) \). We assume that \( W(t) \) satisfies the following properties:

(P1) \( \|W_c\| > 0 \)
(P2) \( 0 < \|F(t)\| \leq F_{max} \)

**Theorem 1.2.** Consider the estimation error dynamics (1.19) with \( W(t) = W_c + F(t) \).

\[ \dot{\varepsilon} = A_\theta \varepsilon + \Phi(\varepsilon, \bar{u}) + W_c + F(t) \quad (1.36) \]

Then, \( r \) is strictly greater than zero if the following inequality is satisfied:

\[ \frac{|\omega_c|}{\theta^n + \gamma \sqrt{H(\theta)}} - \frac{\lambda_{\text{max}}(S_{\theta})}{\lambda_{\text{min}}(S_{\theta})^{3/2}} \left( \frac{2F_{max}}{\theta - 2\gamma} \right) > 0 \quad (1.37) \]

where \( \lambda_{\text{max}}(S_{\theta}) \) and \( \lambda(S_{\theta}) \) are the largest and smallest eigenvalues of matrix \( S_{\theta} \).

Proof. Let us define \( z = \varepsilon - \varepsilon_s \), then, from (1.36) and (1.30) we have

\[ \dot{z} = A_\theta z + \Phi(z, \bar{u}) + F(t) \quad (1.38) \]

where we have used the fact that
\[ \Phi(\varepsilon, \bar{u}) - \Phi(\varepsilon_s, \bar{u}) = \Phi(z + \varepsilon_s, \bar{u}) - \Phi(\varepsilon_s, \bar{u}) = \varphi(\xi + z + \varepsilon_s, \bar{u}) - \varphi(\xi, \bar{u}) = \varphi(z + \xi + \varepsilon_s, \bar{u}) - \varphi(\xi + \varepsilon_s, \bar{u}) = \Phi(z, \bar{u}) \]

The above dynamics is analyzed using the Lyapunov function candidate \( V = z^T S_\theta z \); then

\[
\dot{V} = z^T S_\theta \dot{z} + z^T S_{\theta} \dot{z} = [z^T A_\theta^T + \Phi^T(z, \bar{u}) + F^T(t)] S_\theta z + z^T S_\theta [A_\theta z + \Phi(z, \bar{u}) + F(t)]
\]

Using \( A_\theta = (A - S_\theta^{-1} C^T C) \) and \( \theta S_\theta + A^T S_\theta + S_\theta = C^T C \) we have

\[
\dot{V} = z^T (A^T S_\theta + S_\theta A - C^T C) z - z^T C^T C z + 2\Phi^T(z, \bar{u}) S_\theta z + 2F^T(t) S_\theta z
\]

\[
= -\theta z^T S_\theta z - z^T C^T C z + 2\Phi^T(z, \bar{u}) S_\theta z + 2F^T(t) S_\theta z
\]

Since \( \Phi(z, \bar{u}) \) is differentiable then

\[ \|\Phi^T(z, \bar{u}) S_\theta z\| \leq \gamma \|z\| S_\theta \]

Moreover \( \|F^T(t) S_\theta z\| \leq F_{\text{max}} \|z\| S_\theta \). Then, \( \dot{V} \) is upper bounded as follows:

\[
\dot{V} \leq -\theta \|z\| S_\theta + 2\gamma \|z\| S_\theta + 2F_{\text{max}} \|z\| S_\theta
\]

\[
\leq -(\theta - 2\gamma) \|z\| S_\theta + 2F_{\text{max}} \|z\| S_\theta
\]

Since \( \lambda_{\text{min}}(S_\theta) \|z\|^2 \leq \|z\| S_\theta \leq \lambda_{\text{max}}(S_\theta) \|z\|^2 \) then

\[
\dot{V} \leq -(\theta - 2\gamma) \lambda_{\text{min}}(S_\theta) \|z\|^2 + 2F_{\text{max}} \sqrt{\lambda_{\text{max}}(S_\theta)} \|z\| \tag{1.39}
\]

By applying the UUB theorem [2] we have that \( z \) is bounded uniformly and converges to the compact set \( B_R = \{z \ | \|z\| \leq R \} \) where

\[
R = \sqrt{\frac{\lambda_{\text{max}}(S_\theta)}{\lambda_{\text{min}}(S_\theta)} \left( \frac{2F_{\text{max}} \sqrt{\lambda_{\text{max}}(S_\theta)}}{(\theta - 2\gamma) \lambda_{\text{min}}(S_\theta)} \right) = \frac{\lambda_{\text{max}}(S_\theta)}{\left[ \lambda_{\text{min}}(S_\theta) \right]^{3/2} \left( \frac{2F_{\text{max}}}{\theta - 2\gamma} \right) } \tag{1.40}
\]

Then

\[
\|z\| \leq R = \frac{\lambda_{\text{max}}(S_\theta)}{[\lambda_{\text{min}}(S_\theta)]^{3/2} \left( \frac{2F_{\text{max}}}{\theta - 2\gamma} \right) } \]

Now, since \( z = \varepsilon - \varepsilon_s \), then \( \|\varepsilon - \varepsilon_s\| \leq R \) from which we have

\[
|\varepsilon_1| - |\varepsilon_1| \leq |\varepsilon_1 - \varepsilon_1| \leq R
\]

i.e.

\[
|r_s| - R \leq |\varepsilon_1| \tag{1.41}
\]
where we have used the fact that \( r = \varepsilon_1 \) and \( r_s = \varepsilon_{s1} \). From (1.32) it follows that

\[
|r_s| \geq \frac{|w_c|}{\theta^n + \gamma \sqrt{H(\theta)}}
\]  

(1.42)

Then, replacing (1.40) and (1.42) into (1.41), we obtain

\[
|r_s| - R > \left[ \frac{|w_c|}{\theta^n + \gamma \sqrt{H(\theta)}} - \frac{\lambda_{\max}(S_{\theta})}{[\lambda_{\min}(S_{\theta})]^{3/2}} \left( \frac{2F_{\max}}{\theta - 2\gamma} \right) \right]
\]

Finally, if inequality (1.37) is satisfied we conclude that \( r \geq |r_s| - R > 0 \). □

From the above result it is clear that the constant part \( W_c \) of the fault must dominate over the time-varying part \( F(t) \) for a nonzero residual \( r \) as will be shown in the next section.

1.5 Numerical example

The following example was developed using the MatLab-Simulink® program. Let us consider the following electromechanical positioning system. A bar with centre of mass \( l \) and mass \( m \) is coupled at one of its ends to the shaft of a DC motor. The bar moves in a vertical plane and its rotation is damped through a mechanical damper coupled directly to the motor shaft. A set of springs is attached to the bar in order to add stiffness to the positioning system. Dynamics of the aforementioned system is given by the following second-order equation:

\[
J \ddot{\theta} + f \dot{\theta} + \zeta \theta + mgl \sin(\theta) = k_u u
\]  

(1.43)

where \( \theta \) is the angular displacement of the motor shaft and \( u \) is a voltage applied to the DC motor armature through a power electronic amplifier of gain \( k_u \). \( J \) is the joint inertia of the bar and the motor armature, both referred to the motor axis of rotation. \( f \) is the coefficient of viscous damping of the mechanical damper and \( \zeta \) is the springs stiffness. \( g \) is the gravity constant. Using the change of coordinates \( \xi_1 = \theta \) and \( \xi_2 = \dot{\theta} \), system (1.43) is written as

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -b\xi_2 - c\xi_1 - k_n \sin(\xi_1) + k_g u + w(t)
\end{align*}
\]  

(1.44)

where \( b = f/J, c = \zeta/J, k_n = mgl/J \) and \( k_g = k_u/J \). The term \( w(t) \) accounts for the fault in the system. Note that (1.44) is in the GOCF. Numerical values for the system parameters were \( b = 1, c = 0.4, k_1 = 0.1 \) and \( k_g = 1 \).

The nonlinear term in the GOCF is \( -L_0(\xi, \bar{u}) = -b\xi_2 - c\xi_1 - k_n \sin(\xi_1) + k_g u \). Then, using \( \bar{L}_0(e, \bar{u}) = -L_0(\xi + e, \bar{u}) + L_0(\xi, \bar{u}) \) and \( \Phi(e, \bar{u}) = B\bar{L}_0(e, \bar{u}) \) it is not difficult to show that
where \( \varepsilon = \text{col}(\varepsilon_1 \quad \varepsilon_2) \). Then \( \gamma \geq 2 \) and we have taken \( \gamma = 2 \). An exponential nonlinear observer for (1.44) is given by the following differential equations:

\[
\begin{align*}
\dot{\hat{\xi}_1} &= \hat{\xi}_2 - 2\theta (\hat{\xi}_1 - \xi_1) \\
\dot{\hat{\xi}_2} &= -b\dot{\hat{\xi}_2} - c\hat{\xi}_1 - k_n\sin(\hat{\xi}_1) + k_gu - \theta^2(\hat{\xi}_1 - \xi_1)
\end{align*}
\] (1.45)

where we have used

\[
S^{-1}_\theta = \begin{pmatrix} 2\theta & \theta^2 \\ \theta^2 & \theta^3 \end{pmatrix}
\]

\[
S^{-1}_\theta C^T C = \begin{pmatrix} 2\theta & 0 \\ \theta^2 & 0 \end{pmatrix}
\]

since \( n = 2 \). Then, the residual is given by \( r(t) = \xi_1 - \hat{\xi}_1 \). Since the value for the observer gain must fulfil \( \theta \geq 2\gamma \), we choose \( \theta = 6 \). The positioning system is driven by a sinusoidal voltage \( u = A\sin(\omega t) \) with \( A = 0.5 \) V and \( u = 0.25 \) rad. For the constant fault case we consider a constant voltage produced by a malfunctioning of the power electronics driving the positioning system. Then \( w_c = 1 \) V. The fault occurs at 20 s. **Figure 1.1** shows the time evolution of the angular position and **figure 1.2** depicts the behavior of the residual \( r(t) \) showing clearly a value different from zero. The steady state value of \( |r(t)| \) was \( r_s = 0.02066 \). Inequality (1.32) for \( n = 2 \) may be written as

\[
\frac{|w_c|}{\theta^2 + \gamma\sqrt{1 + 4\theta^2}} \leq |r_s| \quad (1.46)
\]

Computing the left-hand side of (1.46) gives 0.01664. From the above it is clear that (1.46) is satisfied for the steady state value \( |r_s| \).

For the time-varying case a time-varying voltage with an amplitude of 0.001 V is added to the constant fault \( w_c \). Fault also occurs at 20 s. **Figures 1.3 and 1.4** show the behavior of the angular position \( \xi_1 \) and the residual. Note that \( r(t) \) is different from zero. In this case, inequality (1.37) is fulfilled with

\[
\frac{|w_c|}{\theta^2 + \gamma\sqrt{1 + 4\theta^2}} - \frac{\lambda_{\max}(S_\theta)}{[\lambda_{\min}(S_\theta)]^{3/2}} \left( \frac{2F_{\max}}{\theta - 2\gamma} \right) = 0.0008751 > 0
\]

where

\[
\lambda_{\max}(S_\theta) = \frac{\theta}{2} \left( 2 + \theta^2 + \sqrt{4 + \theta^4} \right)
\]

\[
\lambda_{\min}(S_\theta) = \frac{\theta}{2} \left( 2 + \theta^2 - \sqrt{4 + \theta^4} \right)
\]
1.5 Numerical example

Fig. 1.1 Time evolution of the angular position for the constant fault case.

Fig. 1.2 Behavior of the residual $r(t)$ for the constant fault case.
Fig. 1.3 Time evolution of the angular position for the time-varying fault case.

Fig. 1.4 Behavior of the residual $r(t)$ for the time-varying fault case.
References